ANALYTIC COMBINATORICS OF CHORD AND HYPERCHORD DIAGRAMS WITH k CROSSINGS

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Abstract. Using methods from Analytic Combinatorics, we study the families of perfect matchings, partitions, chord diagrams, and hyperchord diagrams on a disk with a prescribed number of crossings. For each family, we express the generating function of the configurations with exactly k crossings as a rational function of the generating function of crossing-free configurations. Using these expressions, we study the singular behavior of these generating functions and derive asymptotic results on the counting sequences of the configurations with precisely k crossings. Limiting distributions and random generators are also studied.

Keywords. Quasi-Planar Configurations – Chord Diagrams – Analytic Combinatorics – Generating Functions

MSC Classes. 05A15 – 05A16 – 05C30 – 05C10

1. Introduction

1.1. **Nearly-planar chord diagrams.** Let V be a set of n labeled points on the unit circle. A *chord diagram* on V is a set of chords between points of V. We say that two chords *cross* when their relative interior intersect. The *crossing graph* of a chord diagram is the graph with a vertex for each chord and an edge between any two crossing chords.

The enumeration properties of crossing-free (or planar) chord diagrams have been widely studied in the literature, see in particular the results of P. Flajolet and M. Noy in [6]. From the work of J. Touchard [21] and J. Riordan [19], we know a remarkable explicit formula for the distribution of crossings among all perfect matchings, which was exploited in [7] to derive, among other parameters, the limit distribution of the number of crossings for matchings with many chords.

A more recent trend studies chord diagrams with some but restricted crossings. The several ways to restrict their crossings lead to various interesting notions of *nearly-planar chord diagrams*. Among others, it is interesting to study chord diagrams

(1) with at most k crossings, or
(2) with no $(k+2)$-crossing (meaning $k+2$ pairwise crossing edges), or
(3) where each chord crosses at most k other chords, or
(4) which become crossing-free when removing at most k well-chosen chords.

Note that these conditions are natural restrictions on the crossing graphs of the chord diagrams. Namely, the corresponding crossing graphs have respectively (1) at most k edges, (2) no $(k+2)$-clique, (3) vertex degree at most k, and (4) a vertex cover of size k. For $k=0$, all these conditions coincide and lead to crossing-free chord diagrams. Other natural restrictions on their crossing graphs can lead to other interesting notions of nearly-planar chord diagrams.

Families of $(k+2)$-crossing-free chord diagrams have been studied in recent literature. On the one hand, $(k+2)$-crossing-free matchings (as well as their $(k+2)$-nesting-free counterparts) were enumerated in [4]. On the other hand, maximal $(k+2)$-crossing-free chord diagrams, also called $(k+1)$-triangulations, were introduced in [2], studied in [15, 18], and enumerated in [12, 20], among others. As far as we know, Conditions (1), (3) and (4), as well as other natural notions of nearly-planar chord diagrams, still remain to be studied in details. We focus in this paper on the Analytic Combinatorics of chord configurations under Condition (1).

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1.2. **Rationality of generating functions.** In this paper, we study enumeration and asymptotic properties for different families of configurations: chord diagrams, hyperchord diagrams (even with restricted hyperchord sizes), perfect matchings, partitions (even with restricted block sizes). Examples of these configurations are represented in Figure 1.

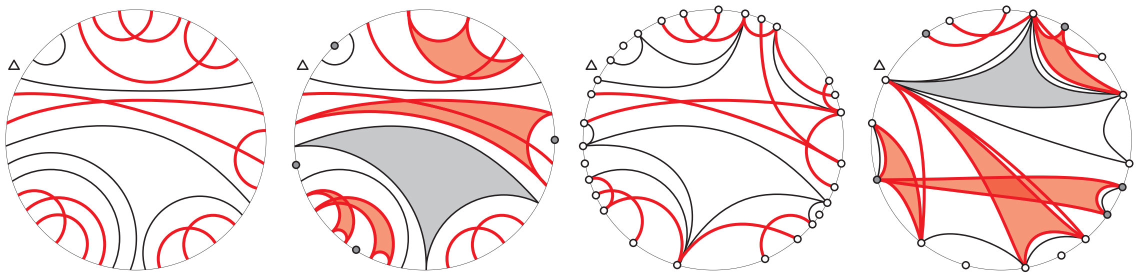

FIGURE 1. The four families of (hyper)chord configurations studied in this paper. Their cores are highlighted in bold red.

Let $\mathcal{C}$ denote one of these families of configurations. For enumeration purposes, we consider the configurations of $\mathcal{C}$ combinatorially: in each configuration we insert a root between two consecutive vertices, and we consider two rooted configurations C and C' of $\mathcal{C}$ as equivalent if there is a continuous bijective automorphism of the circle which sends the root, the vertices, and the (hyper)chords of C to that of C'. We focus on three parameters of the configurations of $\mathcal{C}$: their number n of vertices, their number m of (hyper)chords, and their number k of crossings. Note that for hyperchord diagrams and partitions, we count all crossings involving two chords contained in two distinct hyperchords. Moreover, we can assume that no three chords cross at the same point, so that there is no ambiguity on whether or not we count crossings with multiplicity. We denote by $\mathcal{C}(n, m, k)$ the set of configurations in $\mathcal{C}$ with n vertices, m (hyper)chords and k crossings, and we let

$$\mathbf{C}(x, y, z) := \sum_{n,m,k\in\mathbb{N}} |\mathcal{C}(n, m, k)|\, x^n y^m z^k$$

denote the generating function of $\mathcal{C}$, and

$$\mathbf{C}_k(x, y) := \sum_{n,m\in\mathbb{N}} |\mathcal{C}(n, m, k)|\, x^n y^m = [z^k]\, \mathbf{C}(x, y, z)$$

denote the generating function of the configurations in $\mathcal{C}$ with precisely k crossings. Our first result concerns the rationality of the latter generating function.

Theorem 1.1. *The generating function $\mathbf{C}_k(x, y)$ of configurations in $\mathcal{C}$ with exactly k crossings is a rational function of the generating function $\mathbf{C}_0(x, y)$ of planar configurations in $\mathcal{C}$ and of the variables x and y.*

The idea behind this result is to confine crossings of the configurations of $\mathcal{C}$ to finite subconfigurations. Namely, we define the *core configuration* $C^\star$ of a configuration $C \in \mathcal{C}$ to be the subconfiguration formed by all (hyper)chords of C containing at least one crossing. The key observation is that

(i) there are only finitely many core configurations with k crossings, and
(ii) all configurations of $\mathcal{C}$ with k crossings can be constructed from their core configuration inserting crossing-free subconfigurations in the remaining regions.

This translates in the language of generating functions to a rational expression of $\mathbf{C}_k(x, y)$ in terms of $\mathbf{C}_0(x, y)$ and its successive derivatives with respect to x, which in turn are rational in $\mathbf{C}_0(x, y)$ and the variables x and y. For certain families mentioned above, the dependence in y can even be eliminated, obtaining rational functions in $\mathbf{C}_0(x, y)$ and x .

Similar decomposition ideas were used for example by E. Wright in his study of graphs with fixed excess [22, 23], or more recently by G. Chapuy, M. Marcus, G. Schaeffer in their enumeration of unicellular maps on surfaces [3].

Note that Theorem 1.1 extends a specific result of M. Bóna [1] who proved that the generating function of the partitions with k crossings is a rational function of the generating function of the Catalan numbers. We note that his method was slightly different. The advantage of our decomposition scheme is to be sufficiently elementary and general to apply to the different families of configurations mentioned above.

1.3. **Asymptotic analysis and random generation.** From the rational expression of the generating function $\mathbf{C}_k(x,y)$ in terms of $\mathbf{C}_0(x,y)$, we can extract the asymptotic behavior of configurations in $\mathcal{C}$ with k crossings.

Theorem 1.2. *For $k \geq 1$, the number of configurations in $\mathcal{C}$ with k crossings and n vertices is*

$$[x^n]\,\mathbf{C}_k(x,1) \underset{n\to\infty}{=} \Lambda\, n^{\alpha}\, \rho^{-n}\,(1+o(1)),$$

for certain constants $\Lambda, \alpha, \rho \in \mathbb{R}$ depending on the family $\mathcal{C}$ and on the parameter k. See Table 1.

family	constant Λ	exponent α	singularity ρ^{-1}	Prop.
matchings[1]	$\dfrac{\sqrt{2}\,(2k-3)!!}{4^{k-1}\,k!\,\Gamma\big(k-\frac{1}{2}\big)}$	$k-\dfrac{3}{2}$	2	2.11
partitions[2]	$\dfrac{(2k-3)!!}{2^{3k-1}\,k!\,\Gamma\big(k-\frac{1}{2}\big)}$	$k-\dfrac{3}{2}$	4	2.21
chord diagrams	$\dfrac{\big(-2+3\sqrt{2}\big)^{3k}\sqrt{-140+99\sqrt{2}}\,(2k-3)!!}{2^{3k+1}\,(3-4\sqrt{2})^{k-1}\,k!\,\Gamma(k-\frac{1}{2})}$	$k-\dfrac{3}{2}$	$6+4\sqrt{2}$	3.9
hyperchord diagrams[2,3]	$\simeq \dfrac{1.034^{3k}\,0.003655\,(2k-3)!!}{0.03078^{k-1}\,k!\,\Gamma(k-\frac{1}{2})}$	$k-\dfrac{3}{2}$	$\simeq 64.97$	3.15

TABLE 1. The values of Λ, α and ρ in the asymptoptic estimate of Theorem 1.2 for different families of chord diagrams.

Theorem 1.2 and Table 1 already raise the following remarks:

(i) The position of the singularity of the generating function $\mathbf{C}_k(x,y)$ always arises from that of the corresponding planar family $\mathbf{C}_0(x,y)$. The values of these singularities are very easy to compute for matchings and partitions, but more involved for chord and hyperchord diagrams and for partitions or diagrams with restricted block sizes.

(ii) Although the exponent α seems to always equal $k-\frac{3}{2}$ as in Table 1, this is not true in general. This exponent is dictated by the number of core configurations in $\mathcal{C}$ maximizing a certain functional (see Sections 2.4, 2.6, 3.4, and 3.7). Families of configurations with restricted block sizes can have different exponents, see Sections 2.6 and 3.7.

(iii) Although Theorem 1.2 seems generic, the different families of configurations studied in this paper require different techniques for their asymptotic analysis. Certain methods used for the analysis are elementary, but some other are more complicated machinery borrowed from Analytic Combinatorics [9].

As another relevant application of the rational expression of the generating function $\mathbf{C}_k(x,y)$ from Theorem 1.1, we obtain random generation schemes for the configurations in $\mathcal{C}$ with precisely k crossings, using the methods developed in [5].

[1]The asymptotic estimate for the number of matchings with n vertices is obviously only valid when n is even.

[2]For partitions with restricted block sizes and for hyperchord diagrams with restricted hyperchord sizes, the values of Λ, α and ρ are more involved. We refer to Propositions 2.20 and 3.21 for precise statements.

[3]The expression of ρ^{-1} and Λ for hyperchord diagrams is obtained from approximations of roots of polynomials, and approximate evaluations of analytic functions. Details can be found in Propositions 3.13 and 3.15.

1.4. **Overview.** The paper is organized as follows. In Section 2, we study in full details the case of perfect matchings with k crossings, since we believe that their analysis already illustrates the method and its ramifications, while remaining technically elementary. We define core matchings in Sections 2.1, obtain an expression of the generating function of matchings with k crossings in Section 2.2, study its asymptotic behavior in Sections 2.3 and 2.4, and discuss random generation of matchings with k crossings in Section 2.5. We extend these results to partitions (even with restricted block sizes) in Section 2.6.

In Section 3, we apply the same method to deal with chord diagrams and hyperchord diagrams (even with restricted hyperchord sizes). Although we apply a similar decomposition, the results and analysis are slightly more technical, in particular since the generating functions of crossing-free chord and hyperchord diagrams are not as simple as for matchings and partitions.

Throughout this paper, we use language and basic results of *Analytic Combinatorics*. We refer to the book of P. Flajolet and R. Sedgewick [9] for a detailed presentation of this area. We also mention that a longer preliminary version of the present paper, with more emphasis on examples and computational issues can be found in [17].

2. Perfect matchings and partitions

In this section, we consider the family $\mathcal{M}$ of perfect matchings with endpoints on the unit circle. Each perfect matching M of $\mathcal{M}$ is *rooted*: we mark (with the symbol $\triangle$) an arc of the circle between two endpoints of M, or equivalently, we label the vertices of M counterclockwise starting just after the mark $\triangle$. Although it is equivalent to considering matchings of $[n]$, the representation on the disk suits better for the presentation of our results.

Let $\mathcal{M}(n,k)$ denote the set of matchings in $\mathcal{M}$ with n vertices and k crossings. We denote by

$$\mathbf{M}(x,z) := \sum_{n,k\in\mathbb{N}} |\mathcal{M}(n,k)|\, x^n z^k$$

the generating function of $\mathcal{M}$ where x encodes the number of vertices and z the number of crossings. Observe that we do not encode here the number of chords since it is just half of the number of vertices. We want to study the generating function

$$\mathbf{M}_k(x) := [z^k]\, \mathbf{M}(x,z)$$

of perfect matchings with exactly k crossings.

Example 2.1. The generating function of crossing-free perfect matchings satisfies the functional equation $\mathbf{M}_0(x) = 1 + x^2\, \mathbf{M}_0(x)^2$, leading to the expression

$$\mathbf{M}_0(x) = \frac{1-\sqrt{1-4x^2}}{2x^2} = \sum_{m\in\mathbb{N}} \frac{1}{m+1}\binom{2m}{m} x^{2m} = \sum_{m\in\mathbb{N}} C_m\, x^{2m},$$

where $C_m := \frac{1}{m+1}\binom{2m}{m}$ denotes the m^{th} Catalan number. The asymptotic behavior of the number of crossing-free perfect matchings is thus given by

$$[x^{2m}]\, \mathbf{M}_0(x) = C_m \underset{m\to\infty}{=} \frac{1}{\Gamma\big(\frac{1}{2}\big)}\, n^{-\frac{3}{2}}\, 4^n\, (1+o(1)) = \frac{1}{\sqrt{\pi}}\, n^{-\frac{3}{2}}\, 4^n\, (1+o(1)).$$

The goal of this paper is to go beyond crossing-free objects. We thus assume that $k \geq 1$.

2.1. **Core matchings.** Let M be a perfect matching with some crossings. Our goal is to isolate the contribution of the chords involved in crossings from that of the chords with no crossings.

Definition 2.2. *A core matching is a perfect matching where each chord is involved in a crossing. It is a k-core matching if it has exactly k crossings. The core $M^\star$ of a perfect matching M is the submatching of M formed by all its chords involved in at least one crossing.*

Let K be a core matching. We let $n(K)$ denote its number of vertices and $k(K)$ denote its number of crossings. We call *regions* of K the connected components of the complement of K in the unit disk. A region has i *boundary arcs* if its intersection with the unit circle has i connected arcs. We let $n_i(K)$ denote the number of regions of K with i boundary arcs, and we set $\mathbf{n}(K) := (n_i(K))_{i\in[k]}$. Note that $n(K) = \sum_i i n_i(K)$.

Since a crossing only involves 2 chords, a k-core matching can have at most $2k$ chords. This immediately implies the following crucial observation.

Lemma 2.3. *There are only finitely many k-core matchings.*

The k-core matchings will play a central role in the analysis of the generating function $\mathbf{M}_k(x)$. Hence, we encapsulate the enumerative information of these objects into a formal polynomial in several variables.

Definition 2.4. *We encode the finite list of all possible k-core matchings K and their parameters $n(K)$ and $\mathbf{n}(K) := (n_i(K))_{i\in[k]}$ in the k-core matching polynomial*

$$\mathbf{KM}_k(\mathbf{x}) := \mathbf{KM}_k(x_1, \dots, x_k) := \sum_{\substack{K\ k\text{-core} \\ \text{matching}}} \frac{\mathbf{x}^{\mathbf{n}(K)}}{n(K)} := \sum_{\substack{K\ k\text{-core} \\ \text{matching}}} \frac{1}{n(K)} \prod_{i\in[k]} {x_i}^{n_i(K)}.$$

For later use, we also denote by

$$\mathbf{KM}(\mathbf{x}, z) := \sum_{\substack{K \text{ core} \\ \text{matching}}} \frac{\mathbf{x}^{\mathbf{n}(K)} z^{k(K)}}{n(K)} = \sum_{k\in\mathbb{N}} \mathbf{KM}_k(\mathbf{x})\, z^k$$

the generating function of all core matchings. Note that each core is weighted by the inverse of its number of vertices, both in $\mathbf{KM}_k(\mathbf{x})$ and $\mathbf{KM}(\mathbf{x}, z)$.

2.2. **Generating function of matchings with k crossings.** In this section, we express the generating function $\mathbf{M}_k(x)$ of matchings with k crossings as a rational function of the generating function $\mathbf{M}_0(x)$ of crossing-free matchings, using the k-core matching polynomial $\mathbf{KM}_k(\mathbf{x})$.

We study perfect matchings with k crossings focussing on their k-cores. For this, we consider the following weaker notion of rooting of perfect matchings. We say that a perfect matching with k crossings is *weakly rooted* if we have marked an arc between two consecutive vertices of its k-core. Note that a rooted perfect matching is automatically weakly rooted (the weak root marks the arc of the k-core containing the root of the matching), while a weakly rooted perfect matching corresponds to several rooted perfect matchings. To overtake this technical problem, we use the following immediate rerooting argument.

Lemma 2.5. *Let K be a k-core with $n(K)$ vertices. The number $M_K(n)$ of rooted perfect matchings on n vertices with core K and the number $\bar{M}_K(n)$ of weakly rooted matchings on n vertices with core K are related by $n(K)M_K(n) = n\bar{M}_K(n)$.*

Observe now that we can construct any perfect matching with k crossings by inserting crossing-free submatchings in the regions left by its k-core. From the k-core matching polynomial $\mathbf{KM}_k(\mathbf{x})$, we can therefore derive the following expression of the generating function $\mathbf{M}_k(x)$ of the perfect matchings with k crossings.

Proposition 2.6. *For any $k \geq 1$, the generating function $\mathbf{M}_k(x)$ of the perfect matchings with k crossings is given by*

$$\mathbf{M}_k(x) = x\frac{d}{dx}\mathbf{KM}_k\left(x_i \leftarrow \frac{x^i}{(i-1)!}\frac{d^{i-1}}{dx^{i-1}}\big(x^{i-1}\mathbf{M}_0(x)\big)\right).$$

In particular, $\mathbf{M}_k(x)$ is a rational function of $\mathbf{M}_0(x)$ and x.

Proof. Consider a rooted crossing-free perfect matching M. We say that M is *i-marked* if we have placed $i-1$ additional marks between consecutive vertices of M. Note that we can place more than one mark between two consecutive vertices. Since we have $\binom{n+i-1}{i-1}$ possible ways to place these $(i-1)$ additional marks, the generating function of the i-marked crossing-free perfect matchings is given by

$$\frac{1}{(i-1)!}\frac{d^{i-1}}{dx^{i-1}}\big(x^{i-1}\mathbf{M}_0(x)\big).$$

Consider now a weakly rooted perfect matching M with $k \geq 1$ crossings. We decompose this matching into several submatchings as follows. On the one hand, the core $M^\star$ contains all crossings of M. This core is rooted by the root of M. On the other hand, each region R of $M^\star$ contains a (possibly empty) crossing-free submatching M_R. We root this submatching M_R as follows:

(i) if the root of M points out of the region R, then M_R is just rooted by the root of M;
(ii) otherwise, M_R is rooted on the first boundary arc of $M^\star$ before the root of M in clockwise direction.

Moreover, we place additional marks on the remaining boundary arcs of the complement of R in the unit disk. We thus obtain a rooted i-marked crossing-free submatching M_R in each region R of $M^\star$ with i boundary arcs. Reciprocally, we can reconstruct the weakly rooted perfect matching M from its rooted core $M^\star$ and its rooted i-marked crossing-free submatchings M_R.

By this bijection, we thus obtain the generating function of weakly rooted perfect matchings with k crossings. From this generating function, and by application of Lemma 2.5, we derive the generating function $\mathbf{M}_k(x)$ of rooted perfect matchings with k crossings:

$$\begin{aligned}
\mathbf{M}_k(x) &= \sum_{\substack{K\ k\text{-core}\\ \text{matching}}} \frac{x}{n(K)} \frac{d}{dx} x^{n(K)} \prod_{i\geq 1} \left(\frac{1}{(i-1)!} \frac{d^{i-1}}{dx^{i-1}} \big(x^{i-1} \mathbf{M}_0(x) \big) \right)^{n_i(K)} \\
&= x \frac{d}{dx} \sum_{\substack{K\ k\text{-core}\\ \text{matching}}} \frac{1}{n(K)} \prod_{i\geq 1} \left(\frac{x^i}{(i-1)!} \frac{d^{i-1}}{dx^{i-1}} \big(x^{i-1} \mathbf{M}_0(x) \big) \right)^{n_i(K)} \qquad (1) \\
&= x \frac{d}{dx} \mathbf{KM}_k \left(x_i \leftarrow \frac{x^i}{(i-1)!} \frac{d^{i-1}}{dx^{i-1}} \big(x^{i-1} \mathbf{M}_0(x) \big) \right).
\end{aligned}$$

Since $\mathbf{M}_0(x)$ is given by

$$\mathbf{M}_0(x) = \frac{1-\sqrt{1-4x^2}}{2x^2}$$

and satisfies the functional equation

$$\mathbf{M}_0(x) = 1 + x^2\, \mathbf{M}_0(x)^2,$$

its derivative is rational in $\mathbf{M}_0(x)$ and x. By induction, all its successive derivatives, and therefore $\mathbf{M}_k(x)$, are also rational in $\mathbf{M}_0(x)$ and x. $\square$

Example 2.7. From the 1-core polynomial $\mathbf{KM}_1(\mathbf{x}) = \frac{1}{4}\, {x_1}^4$, we obtain the generating function for matchings with a single crossing

$$\mathbf{M}_1(x) = \frac{x^4\, \mathbf{M}_0(x)^4}{1-2x^2\, \mathbf{M}_0(x)} = \frac{\big(1-\sqrt{1-4x^2}\big)^4}{16x^4\sqrt{1-4x^2}} = x^4 + 6\,x^6 + 28\,x^8 + 120\,x^{10} + 495\,x^{12} + 2002\,x^{14} \dots$$

These coefficients are indexed as Sequence A002694 in the *Sloane's On-Line Encyclopedia of Integer Sequences* [16].

2.3. **Maximal core matchings.** Before establishing asymptotic formulas of the number of perfect matchings with k crossings in Section 2.4, we need to introduce and characterize here certain k-core matchings that we call *maximal*.

Example 2.8. Figure 2 illustrates the first few examples of a family of k-core matchings with $n_k(K) = 1$. Note that, except the first one, these k-core matchings can be rooted in four different (meaning non-equivalent) positions.

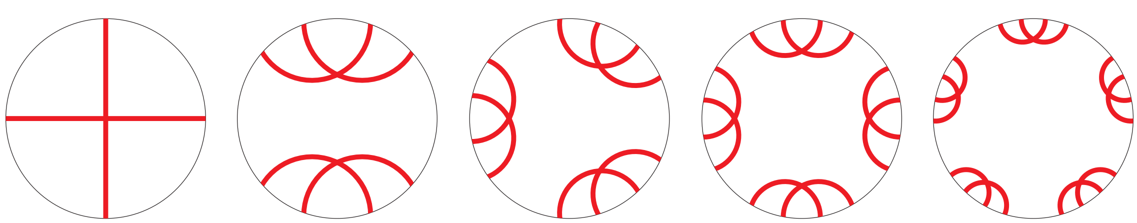

FIGURE 2. Maximal core matchings (unrooted).

Lemma 2.9. *The following assertions are equivalent for an (unrooted) k-core matching K:*

(i) K is one of the k-core matchings presented in Figure 2.
(ii) $n_1(K) = 3k$, $n_k(K) = 1$ and $n_i(K) = 0$ for all other values of i (here, $k \geq 2$).
(iii) K maximizes $n_1(K)$ among all possible k-core matchings (here, $k \geq 3$).
(iv) K maximizes the potential function $\phi(K) \coloneqq \sum_{i>1}(2i-3)\, n_i(K)$ among all k-core matchings.

We call maximal *a k-core matching satisfying these conditions.*

Proof. Assume that $k \geq 2$. The implication (i) $\Longrightarrow$ (ii) is immediate. For the reverse implication, observe that if a region R of K has k boundary arcs, then K has at least, and thus precisely, one crossing between any two consecutive boundary arcs of R. This implies that K is one of the k-core matchings presented in Figure 2.

We now prove that (ii) $\Longleftrightarrow$ (iii) when $k \geq 3$. Observe that $n_1(K) = 4 = 3\,k(K) + 1$ for the unique 1-core matching K, and that $n_1(K) = 6 = 3\,k(K)$ for any 2-core matching K. Given any core matching K with $k \geq 3$ crossings, we now prove by induction on the number of connected components of K that $n_1(K) \leq 3k$, with equality if and only if K satisfies the conditions of (ii). If the crossing graph of K is connected, we have $n_1(K) \leq 2(k+1) < 3k$. Otherwise, we split the unit disk along a region of K with $r > 2$ boundary arcs, and we obtain r core matchings $K_1, \dots, K_r$. Observe that

$$k(K) = \sum_{j\in[r]} k(K_j) \qquad \text{and} \qquad n(K) = \sum_{j\in[r]} n(K_j),$$

where the second equality can be refined to

$$n_1(K) = \sum_{j\in[r]} \big(n_1(K_j) - 1\big), \quad n_r(K) = 1 + \sum_{j\in[r]} n_r(K), \quad \text{and} \quad n_i(K) = \sum_{j\in[r]} n_i(K) \quad \text{for } i \notin \{1, r\}.$$

Let s denote the number of core matchings K_j with $k(K_j) > 1$. For these cores K_j, we have $n_1(K_j) \leq 3k(K_j)$ by induction hypothesis (and by our previous observation on the special case of 2-core matchings). For the other cores K_j, with $k(K_j) = 1$, we have $n_1(K_j) = 4 = 3\,k(K_j) + 1$ as observed earlier. Therefore, we obtain

$$n_1(K) = \sum_{j\in[r]} \big(n_1(K_j) - 1\big) \leq \left(\sum_{j\in[r]} 3k(K_j)\right) - s = 3k(K) - s \leq 3k(K),$$

with equality if and only if $s = 0$. The latter condition is clearly equivalent to (ii).

Using a similar method, we finally prove that (ii) $\Longleftrightarrow$ (iv) when $k \geq 2$. Namely, given a core matching K with $k \geq 2$ crossings, we prove by induction on the number of connected components of K that $\phi(K) \leq 2k - 3$, with equality if and only if K satisfies the conditions of (ii). If K is connected, then $n_i(K) = 0$ for all $i > 1$, and $\phi(K) = 0 < 2k - 3$. Otherwise, we split the unit disk along a region of K with $r > 2$ boundary arcs, and we obtain r core matchings $K_1, \dots, K_r$. Let s denote the number of core matchings K_j with $k(K_j) > 1$. Up to relabeling, we can assume that $K_1, K_2, \dots, K_s$ are the cores K_j with more than 1 crossing. By induction hypothesis, we have for all $j \in [s]$,

$$\sum_{i>1} (2i-3)\, n_i(K_j) \leq 2k(K_j) - 3,$$

and therefore

$$\sum_{j\in[s]} \sum_{i>1} (2i-3)\, n_i(K_j) \leq 2 \sum_{j\in[s]} k(K_j) - 3s.$$

For the core matching K, we therefore obtain

$$\begin{aligned}
\phi(K) = \sum_{i>1}(2i-3)\, n_i(K) &= (2r-3) + \sum_{j\in[s]} \sum_{i>1} (2i-3)\, n_i(K_j) \\
&\leq (2r-3) + 2 \sum_{j\in[s]} k(K_j) - 3s = 2\left(r - s + \sum_{j\in[s]} k(K_j)\right) - 3 - s \\
&= 2\,k(K) - 3 - s \leq 2\,k(K) - 3,
\end{aligned}$$

with equality if and only if $s = 0$, *i.e.* if and only if K satisfies the conditions of (ii) $\square$

2.4. **Asymptotic analysis.** We now describe the asymptotic behavior of the number of perfect matchings with $k \geq 1$ crossings. We start with the asymptotics of perfect matchings with a single crossing, which can be worked out from the explicit expression obtained in Example 2.7.

Example 2.10. Setting $X_+ := \sqrt{1-2x}$ and $X_- := \sqrt{1+2x}$, we rewrite the expression of the generating function $\mathbf{M}_1(x)$ obtained in Example 2.7 as

$$\mathbf{M}_1(x) = \frac{(1 - X_+ X_-)^4}{16x^4 X_+ X_-}.$$

Direct expansions around the singularities $x = \pm\frac{1}{2}$ of X_+ and X_- give

$$\mathbf{M}_1(x) \underset{x\sim\frac{1}{2}}{=} \frac{1}{\sqrt{2}} {X_+}^{-1} + O(1) \qquad \text{and} \qquad \mathbf{M}_1(x) \underset{x\sim-\frac{1}{2}}{=} \frac{1}{\sqrt{2}} {X_-}^{-1} + O(1).$$

Applying the Transfer Theorem for singularity analysis [8, 9], we obtain:

$$[x^n]\,\mathbf{M}_1(x) \underset{n\to\infty}{=} \frac{1}{\sqrt{2}\,\Gamma\big(\frac{1}{2}\big)}\, n^{-\frac{1}{2}}\, (2^n + (-2)^n)(1 + o(1)),$$

Writing this expression for $n = 2m$, we get the final estimate

$$[x^{2m}]\,\mathbf{M}_1(x) \underset{m\to\infty}{=} \frac{1}{\Gamma\big(\frac{1}{2}\big)}\, m^{-\frac{1}{2}}\, 4^m\, (1 + o(1)).$$

The analysis is more involved for general values of k. The method consists in studying the asymptotic behavior of $\mathbf{M}_0(x)$ and of all its derivatives around their minimal singularities, and to exploit the rational expression of $\mathbf{M}_k(x)$ in terms of $\mathbf{M}_0(x)$ and x given in Proposition 2.6. Along the way, we naturally study which k-cores have the main asymptotic contributions. In fact, the potential function studied in Section 2.3 will naturally show up in the analysis, and the main contribution to the number of perfect matchings with k crossings and n vertices will asymptotically arise from the maximal k-core matchings (observe that in the special case $k = 1$, the unique 1-core is maximal). We obtain the following asymptotic estimates.

Proposition 2.11. *For any $k \geq 1$, the number of perfect matchings with k crossings and $n = 2m$ vertices is*

$$[x^{2m}]\,\mathbf{M}_k(x) \underset{m\to\infty}{=} \frac{(2k-3)!!}{2^{k-1}\,k!\,\Gamma\big(k-\frac{1}{2}\big)}\, m^{k-\frac{3}{2}}\, 4^m\, (1 + o(1)),$$

where $(2k-3)!! := (2k-3)\cdot(2k-5)\cdots 3\cdot 1$.

Proof. The result follows from Example 2.10 when $k = 1$. We can thus assume that $k \geq 2$. We first study the asymptotic behavior of $\mathbf{M}_0(x)$ and of all its derivatives around their minimal singularities. The generating function $\mathbf{M}_0(x)$ defines an analytic function around the origin. Its dominant singularities are located at $x = \pm\frac{1}{2}$. Denoting by $X_+ := \sqrt{1-2x}$ and $X_- := \sqrt{1+2x}$, the Puiseux's expansions of $\mathbf{M}_0(x)$ around $x = \frac{1}{2}$ and $x = -\frac{1}{2}$ are

$$\mathbf{M}_0(x) \underset{x\sim\frac{1}{2}}{=} 2 - 2\sqrt{2}\,X_+ + O\left({X_+}^2\right) \qquad \text{and} \qquad \mathbf{M}_0(x) \underset{x\sim-\frac{1}{2}}{=} 2 - 2\sqrt{2}\,X_- + O\left({X_-}^2\right),$$

valid in a domain dented at $x = 1/2$ and $x = -1/2$, respectively (see [9]). Consequently,

$$\frac{d}{dx}\mathbf{M}_0(x) \underset{x\sim\frac{1}{2}}{=} 2\sqrt{2}\,{X_+}^{-1} + O(1) \qquad \text{and} \qquad \frac{d}{dx}\mathbf{M}_0(x) \underset{x\sim-\frac{1}{2}}{=} -2\sqrt{2}\,{X_-}^{-1} + O(1),$$

and for $i > 1$, the i^{th} derivative of $\mathbf{M}_0(x)$ has singular expansion around $x = \pm\frac{1}{2}$

$$\frac{d^i}{dx^i}\mathbf{M}_0(x) \underset{x\sim\frac{1}{2}}{=} 2\sqrt{2}\,(2i-3)!!\,{X_+}^{1-2i} + O\left({X_+}^{2-2i}\right),$$

$$\frac{d^i}{dx^i}\mathbf{M}_0(x) \underset{x\sim-\frac{1}{2}}{=} (-1)^i\, 2\sqrt{2}\,(2i-3)!!\,{X_-}^{1-2i} + O\left({X_-}^{2-2i}\right),$$

where $(2i-3)!! := (2i-3)\cdot(2i-5)\cdots 3\cdot 1$. These expansions are also valid in a dented domain at $x = \frac{1}{2}$ and $x = -\frac{1}{2}$, respectively.

We now exploit the expression of the generating function $\mathbf{M}_k(x)$ given by Equation (1) in the proof of Proposition 2.6. The dominant singularities of $\mathbf{M}_k(x)$ are located at $x = \pm\frac{1}{2}$. We provide the full analysis around $x = \frac{1}{2}$, the computation for $x = -\frac{1}{2}$ being similar. For conciseness in the following expressions, we set by convention $(-1)!! = 1$. We therefore obtain:

$$\begin{aligned}
\mathbf{M}_k(x) &= x\frac{d}{dx}\sum_{\substack{K\ k\text{-core}\\ \text{matching}}}\frac{1}{n(K)}\prod_{i\geq 1}\left(\frac{x^i}{(i-1)!}\frac{d^{i-1}}{dx^{i-1}}\left(x^{i-1}\mathbf{M}_0(x)\right)\right)^{n_i(K)}\\
&\underset{x\sim\frac{1}{2}}{=} \frac{1}{2}\frac{d}{dx}\sum_{\substack{K\ k\text{-core}\\ \text{matching}}}\frac{1}{n(K)}\prod_{i\geq 1}\left(\frac{1}{2^{2i-1}\,(i-1)!}\frac{d^{i-1}}{dx^{i-1}}\mathbf{M}_0(x)\right)^{n_i(K)}\\
&\underset{x\sim\frac{1}{2}}{=} \frac{1}{2}\frac{d}{dx}\sum_{\substack{K\ k\text{-core}\\ \text{matching}}}\frac{1}{n(K)}\prod_{i>1}\left(-\frac{\sqrt{2}\,(2i-5)!!}{4^{i-1}\,(i-1)!}\,{X_+}^{3-2i}+O\left({X_+}^{4-2i}\right)\right)^{n_i(K)}\\
&\underset{x\sim\frac{1}{2}}{=} \frac{1}{2}\frac{d}{dx}\sum_{\substack{K\ k\text{-core}\\ \text{matching}}}\frac{1}{n(K)}\prod_{i>1}\left(-\frac{\sqrt{2}\,(2i-5)!!}{4^{i-1}\,(i-1)!}\right)^{n_i(K)}{X_+}^{-\phi(K)}+O\left({X_+}^{-\phi(K)+1}\right)\\
&\underset{x\sim\frac{1}{2}}{=} \frac{1}{2}\sum_{\substack{K\ k\text{-core}\\ \text{matching}}}\frac{-\phi(K)}{n(K)}\prod_{i>1}\left(-\frac{\sqrt{2}\,(2i-5)!!}{4^{i-1}\,(i-1)!}\right)^{n_i(K)}{X_+}^{-\phi(K)-2}+O\left({X_+}^{-\phi(K)-1}\right),
\end{aligned}$$

where $\phi(K) := \sum_{i>1}(2i-3)\,n_i(K)$ denotes the potential function studied in Section 2.3. Observe that in order to obtain the third equality, we used the fact that $k>1$, and thus, that there exists k-cores K such that $n_i(K)\neq 0$ when $i>1$. Combining Lemma 2.9 with the Transfer Theorem for singularity analysis [8, 9], we conclude that the main contribution in the asymptotic of the previous sum arises from maximal k-cores, as they maximize the value $2+\phi(K)$. There are exactly four maximal k-cores with $n_1(K)=3k$, $n_k(K)=1$, $n(K)=4k$, and $\phi(K)=2k-3$. Hence,

$$\begin{aligned}
[x^n]\,\mathbf{M}_k(x) &\underset{x\sim\frac{1}{2}}{=} [x^n]\,\frac{1}{2}\sum_{\substack{K\ k\text{-core}\\ \text{matching}}}\frac{-\phi(K)}{n(K)}\prod_{i>1}\left(-\frac{\sqrt{2}\,(2i-5)!!}{4^{i-1}\,(i-1)!}\right)^{n_i(K)}{X_+}^{-\phi(K)-2}+O\left({X_+}^{-\phi(K)-1}\right)\\
&\underset{x\sim\frac{1}{2}}{=} \frac{2\sqrt{2}\,(2k-3)!!}{4^k\,k!}\,[x^n]\,\sqrt{1-2x}^{\,1-2k}+O\left((1-2x)^{1-k}\right)\\
&\underset{n\to\infty}{=} \frac{2\sqrt{2}\,(2k-3)!!}{4^k\,k!\,\Gamma\left(k-\frac{1}{2}\right)}\,n^{k-\frac{3}{2}}\,2^n\,(1+o(1)),
\end{aligned}$$

where the last equality is obtained by an application of the Transfer Theorem for singularity analysis [8, 9].

Finally, we obtain the stated result by adding together the expression obtained when studying $\mathbf{M}_k(x)$ around $x=\frac{1}{2}$ and $x=-\frac{1}{2}$. In fact, one can check that the asymptotic estimate of $[x^n]\,\mathbf{M}_k(x)$ around $x=-\frac{1}{2}$ is the same but with an additional multiplicative constant $(-1)^n$. Consequently, the contribution is equal to 0 when n is odd and the estimate in the statement when $n=2m$ is even. □

2.5. **Random generation.** The composition scheme presented in Proposition 2.6 can also be exploited in order to provide Boltzmann samplers for random generation of perfect matchings with k crossings. Throughout this section we consider a positive real number $\theta<\frac{1}{2}$, which acts as a "control-parameter" for the random sampler (see [5] for further details).

The Boltzmann sampler works in three steps:

(i) We first decide which is the core of our random object.
(ii) Once this core is chosen, we complete the matching by means of non-crossing (and possibly marked) matchings.
(iii) Finally, we place the root of the resulting perfect matching with k crossings.

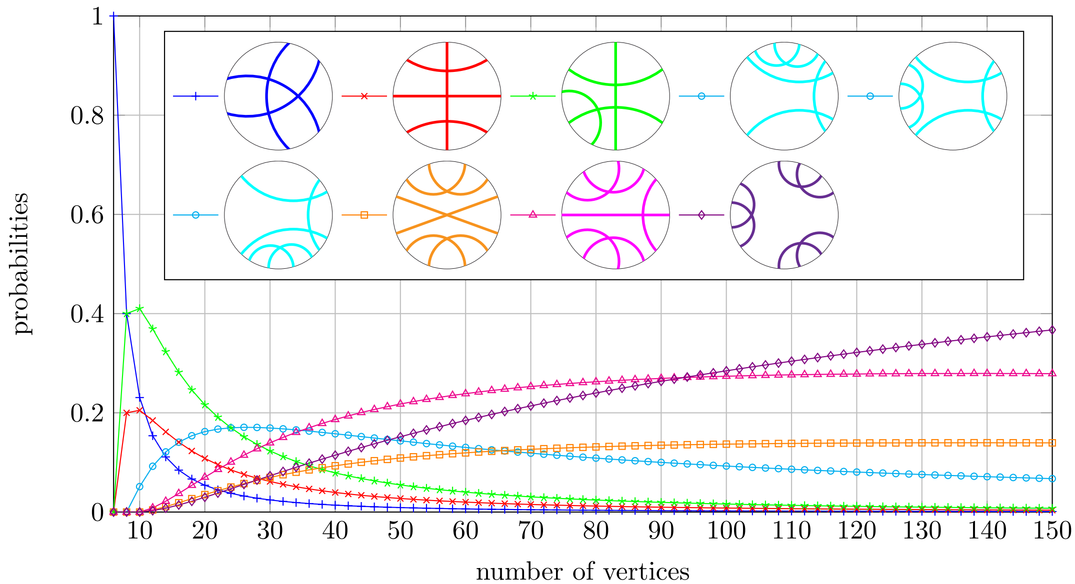

FIGURE 3. Probabilities of appearance of the different 3-core matchings.

We start with the choice of the k-core. For each k-core K, let $\mathbf{M}_K(x)$ denote the generating function of matchings with k crossings and whose k-core is K, where x marks as usual the number of vertices. Note that this generating function is computed as in Proposition 2.6, using only the contribution of the k-core K. Therefore, we have

$$\mathbf{M}_k(x) = \sum_{\substack{K\ k\text{-core} \\ \text{matching}}} \mathbf{M}_K(x)\,.$$

This sum defines a probability distribution in the following way: once fixed the parameter θ, let

$$p_K = \frac{\mathbf{M}_K(\theta)}{\mathbf{M}_k(\theta)}.$$

This set of values defines a Bernoulli distribution $\{p_K\}_{\substack{K\ k\text{-core} \\ \text{matching}}}$, which can be easily simulated.

Remark 2.12. As it has been pointed out in Section 2.4, the main contribution to the enumeration of perfect matchings with k crossings, when the number of vertices is large enough, arises from the ones whose k-core is maximal. Consequently, when θ is close enough to $\frac{1}{2}$, the first step in the random sampling would provide a maximal core with high probability. To illustrate this fact, we have represented in Figure 3 the probability of each possible 3-core for a random perfect matching with 3 crossings.

Once we have fixed the core of the random matching, we continue in the second step filling in its regions with crossing-free perfect matchings. For this purpose it is necessary to start having a procedure to generate crossing-free perfect matchings, namely $\Gamma\mathbf{M}_0(\theta)$. As $\mathbf{M}_0(\theta)$ satisfies the recurrence relation $\mathbf{M}_0(\theta) = 1 + \theta^2\mathbf{M}_0(\theta)^2$, a Boltzmann sampler $\Gamma\mathbf{M}_0(\theta)$ can be defined in the following way. Let $p = \frac{1}{\Gamma\mathbf{M}_0(\theta)}$. Then, using the language of [5],

$$\Gamma\mathbf{M}_0(\theta) := \mathrm{Bern}(p) \longrightarrow \varnothing \mid (\Gamma\mathbf{M}_0(\theta)\ ,\ \bullet - \bullet\ ,\ \Gamma\mathbf{M}_0(\theta)),$$

where $\bullet - \bullet$ means that the Boltzmann sampler is generating a single chord (or equivalently, two vertices in the border of the circle). This Boltzmann sampler is defined when $\theta < \frac{1}{2}$, in which case the defined branching process is subcritical. In such situation the algorithm stops in finite expected time, see [5].

Once this random sampler is performed, we can deal with a term of the form $\frac{d^{i-1}}{dx^{i-1}}x^{i-1}\mathbf{M}_0(\theta)$. Indeed, once a random crossing-free perfect matching $\Gamma\mathbf{M}_0(\theta)$ of size $n(\Gamma\mathbf{M}_0(\theta))$ is generated, there exist

$$\binom{n(\Gamma\mathbf{M}_0(\theta))+i-1}{i-1}$$

i-marked crossing-free perfect matching arising from $\Gamma\mathbf{M}_0(\theta)$. Hence, with uniform probability we can choose one of these i-marked crossing-free perfect matchings. As this argument follows for each choice of i, and $\mathbf{KM}_K(\mathbf{x})$ is a polynomial, we can combine the generator of i-marked crossing-free diagrams with the Boltzmann sampler for the cartesian product of combinatorial classes (recall that we need to provide the substitution $x_i \leftarrow \frac{x^i}{(i-1)!}\frac{d^{i-1}}{dx^{i-1}}\left(x^{i-1}\mathbf{M}_0(x)\right)$).

Finally, we need to apply the root operator, which can be done by means of similar arguments as in the case of i-marked crossing-free diagrams.

Concerning the statistics of the random variable N corresponding to the size of the element generated by means of the previous random sampler, as it is shown in [5], the expected value $\mathbb{E}[N]$ and the variance $\mathbb{V}\mathrm{ar}[N]$ of the random variable N satisfy

$$\mathbb{E}[N] = \theta\,\frac{\mathbf{M}_k'(\theta)}{\mathbf{M}_k(\theta)} \qquad \text{and} \qquad \mathbb{V}\mathrm{ar}[N] = \frac{\theta^2(\mathbf{M}_k''(\theta)\,\mathbf{M}_k(\theta) - \theta\,\mathbf{M}_k'(\theta)^2) + \theta\,\mathbf{M}_k'(\theta)}{\mathbf{M}_k(\theta)^2}.$$

Hence, when θ tends to $\frac{1}{2}$, the expected value of the generated element tends to infinity, and the variance for the expected size also diverges. Consequently, the random variable N is not concentrated around its expected value.

Example 2.13. For perfect matchings with 3 crossings, the expectation $\mathbb{E}[N]$ and the variance $\mathbb{V}\mathrm{ar}[N]$ are given by

θ	0.40	0.45	0.465	0.475	0.48	0.4999
$\mathbb{E}[N]$	17.31	30.66	41.78	56.42	69.14	12508.22
$\sqrt{\mathbb{V}\mathrm{ar}[N]}$	7.69	44.44	109.44	249.83	427.32	$0.7406 \cdot 10^8$

2.6. **Extension to partitions.** We close this section by extending our results from perfect matchings to partitions. We now consider the family $\mathcal{P}$ of partitions of point sets on the unit circle. As before, the partitions are rooted by a mark on an arc between two vertices. A *crossing* between two blocks U, V of a partition P is a pair of crossing chords u_1u_2 and v_1v_2 where $u_1, u_2 \in U$ and $v_1, v_2 \in V$. We count crossings with multiplicity: two blocks U, V cross as many times as the number of such pairs of crossing chords among U and V.

For a non-empty subset S of $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$, we denote by $\mathcal{P}^S$ the family of partitions of point sets on the unit circle, where the cardinality of each block belongs to the set S. For example, matchings are partitions where all blocks have size 2, *i.e.* $\mathcal{M} = \mathcal{P}^{\{2\}}$. Observe that depending on S and k, it is possible that no partition of $\mathcal{P}^S$ has exactly k crossings. For example, since two triangles can have either 0, 4, or 6 crossings, there is no 3-uniform partition (*i.e.* with $S = \{3\}$) with an odd number of crossings.

Applying the same method as in Section 2.2, we obtain an expression of the generating function $\mathbf{P}_k^S(x, y)$ of partitions of $\mathcal{P}^S$ with k crossings in terms of the corresponding k-core partition polynomial

$$\mathbf{KP}_k^S(\mathbf{x}, y) := \sum_{\substack{K\ k\text{-core} \\ \text{partition of } \mathcal{P}^S}} \frac{\mathbf{x}^{\mathbf{n}(K)} y^{m(K)}}{n(K)}.$$

We say that $S \subset \mathbb{N}^*$ is *ultimately periodic* if it can be written as

$$S = A_S \cup \bigcup_{b \in B_S} \{b + up_S \mid u \in \mathbb{N}\}$$

for two finite subsets $A_S, B_S \subset \mathbb{N}^*$ and a period $p_S \in \mathbb{N}^*$.

Proposition 2.14. *For any $k \ge 1$, the generating function $\mathbf{P}_k^S(x,y)$ of partitions with k crossings and where the size of each block belongs to S is given by*

$$\mathbf{P}_k^S(x,y) = x\frac{d}{dx}\mathbf{K}\mathbf{P}_k^S\left(x_i \leftarrow \frac{x^i}{(i-1)!}\frac{d^{i-1}}{dx^{i-1}}\big(x^{i-1}\mathbf{P}_0^S(x,y)\big), y\right).$$

If S is finite or ultimately periodic, then $\mathbf{P}_k^S(x,y)$ is a rational function of $\mathbf{P}_0^S(x,y)$ and x.

Proof. The proof is again similar to that of Proposition 2.6, replacing matchings by partitions of $\mathcal{P}^S$. Again, the difference lies in proving that the successive derivatives of $\mathbf{P}_0^S(x,y)$ and the variable y are all rational functions of $\mathbf{P}_0^S(x,y)$ and x. Splitting a crossing-free partition of $\mathcal{P}^S$ with respect to its block containing its first vertex, we obtain the functional equation

$$\mathbf{P}_0^S(x,y) = 1 + y\sum_{s\in S} x^s\,\mathbf{P}_0^S(x,y)^s.$$

If S is finite or ultimately periodic, we write $S = A_S \cup \bigcup_{b\in B_S}\{b + up_S \mid u \in \mathbb{N}\}$ for finite subsets $A_S, B_S \subset \mathbb{N}^*$ and a period $p_S \in \mathbb{N}^*$, and we can write

$$\sum_{s\in S} t^s = A_S(t) + \frac{B_S(t)}{1-t^{p_S}},$$

where $A_S(t) \coloneqq \sum_{a\in A_S} t^a$ and $B_S(t) \coloneqq \sum_{b\in B_S} t^b$. We thus obtain that

$$\begin{aligned} &(\mathbf{P}_0^S(x,y) - 1 - y\,A_S(x\,\mathbf{P}_0^S(x,y)))(1 - x^{p_S}\,\mathbf{P}_0^S(x,y)^{p_S}) - y\,B_S(x\,\mathbf{P}_0^S(x,y)) = 0 \\ \text{and}\qquad &y = \frac{(\mathbf{P}_0^S(x,y)-1)\big(1 - x^{p_S}\,\mathbf{P}_0^S(x,y)^{p_S}\big)}{A_S(x\,\mathbf{P}_0^S(x,y))\big(1 - x^{p_S}\,\mathbf{P}_0^S(x,y)^{p_S}\big) + B_S(x\,\mathbf{P}_0^S(x,y))}. \end{aligned}$$

Derivating the former functional equation ensures that the successive derivatives of $\mathbf{P}_0^S(x,y)$ are all rational functions of $\mathbf{P}_0^S(x,y)$ and the variables x and y. The latter equation ensures that y itself is rational in $\mathbf{P}_0^S(x,y)$ and x, thus concluding the proof. □

From the expression of $\mathbf{P}_k^S(x,y)$ given in Proposition 2.14, we can extract asymptotic estimates for the number of partitions with k crossings and where the size of each block belongs to S. The difficulty here lies in two distinct aspects:

(i) estimate the minimal singularity ρ_S and describe the singular behavior around ρ_S of the generating function $\mathbf{P}_0^S(x,1)$ of crossing-free partitions of $\mathcal{P}^S$, and
(ii) characterize which k-core partitions of $\mathcal{P}^S$ have the main contribution to the asymptotic.

The first point is discussed in details below in Proposition 2.17. In contrast, we are able to handle the second point only for particular cases. The following constants will be needed in Propositions 2.17 and 2.20.

Definition 2.15. *Given a non-empty subset S of $\mathbb{N}^*$ different from the singleton $\{1\}$, we define τ_S to be the unique positive real number such that*

$$\sum_{s\in S}(s-1){\tau_S}^s = 1.$$

We furthermore define the constants ρ_S, α_S and β_S to be

$$\rho_S \coloneqq \frac{\tau_S}{\sum_{s\in S} s{\tau_S}^s}, \qquad \alpha_S \coloneqq 1 + \sum_{s\in S}{\tau_S}^s, \qquad \textit{and} \qquad \beta_S \coloneqq \sqrt{\frac{2\left(\sum_{s\in S} s{\tau_S}^s\right)^3}{\sum_{s\in S} s(s-1){\tau_S}^s}}.$$

Remark 2.16. Observe that τ_S is indeed well-defined, unique and belongs to $]0,1]$. Indeed the function $\tau \mapsto \sum_{s\in S}(s-1)\tau^s$ is strictly increasing, evaluates to 0 when $\tau = 0$, and is either a power series with radius of convergence 1 (if S is infinite), or a polynomial which evaluates at least to 1 when $\tau = 1$ (if S is finite). Observe also that

$$\rho_S = \frac{\tau_S}{1 + \sum_{s\in S}{\tau_S}^s} \qquad \text{and} \qquad \alpha_S = \frac{\tau_S}{\rho_S},$$

and that these two constants are both positive.

These constants naturally appear in the proof of the following statement, which describes the singular behavior of $\mathbf{P}_0^S(x,1)$ and the asymptotic of its coefficients.

Proposition 2.17. *For any non-empty subset S of $\mathbb{N}^*$ different from the singleton $\{1\}$, the generating function $\mathbf{P}_0^S(x,1)$ satisfies*

$$\mathbf{P}_0^S(x,1) \underset{x\sim\rho_S}{=} \alpha_S - \beta_S\sqrt{1-\frac{x}{\rho_S}} + O\left(1-\frac{x}{\rho_S}\right),$$

in a domain dented at $x=\rho_S$, for the constants ρ_S, α_S and β_S described in Definition 2.15. Therefore, its coefficients satisfy

$$[x^n]\,\mathbf{P}_0^S(x,1) \underset{\substack{n\to\infty\\ \gcd(S)|n}}{=} \frac{\gcd(S)\,\beta_S}{2\sqrt{\pi}}\, n^{-\frac{3}{2}}\, {\rho_S}^{-n}\,(1+o(1))$$

for n multiple of $\gcd(S)$, while $[x^n]\,\mathbf{P}_0^S(x,1)=0$ if n is not a multiple of $\gcd(S)$.

Proof. We apply the theorem of A. Meir and J. Moon [14] on the singular behavior of generating functions defined by a smooth implicit-function schema. As already observed, the generating function $\mathbf{P}_0^S(x,1)$ satisfies the functional equation

$$\mathbf{P}_0^S(x,1) = 1 + \sum_{s\in S} x^s\,\mathbf{P}_0^S(x,1)^s\,.$$

If we set

$$\mathbf{W}(x) \coloneqq \mathbf{P}_0^S(x,1)-1 \qquad \text{and} \qquad \mathbf{G}(x,w) \coloneqq \sum_{s\in S} x^s(w+1)^s,$$

then we obtain a smooth implicit-function schema $\mathbf{W}(x)=\mathbf{G}(x,\mathbf{W}(x))$. Indeed, if we fix

$$u \coloneqq \rho_S = \frac{\tau_S}{\sum_{s\in S} s{\tau_S}^s} \qquad \text{and} \qquad v \coloneqq \alpha_S - 1 = \frac{\tau_S}{\rho_S} - 1 = \sum_{s\in S} {\tau_S}^s,$$

we observe that

$$\mathbf{G}(u,v) = \sum_{s\in S} u^s(v+1)^s = \sum_{s\in S} {\rho_S}^s\left(\frac{\tau_S}{\rho_S}\right)^s = \sum_{s\in S} {\tau_S}^s = v \qquad \text{and}$$

$$\mathbf{G}_w(u,v) = \sum_{s\in S} s u^s(v+1)^{s-1} = \sum_{s\in S} s{\rho_S}^s\left(\frac{\tau_S}{\rho_S}\right)^{s-1} = \frac{\rho_S}{\tau_S}\sum_{s\in S} s{\tau_S}^s = 1.$$

The statement is therefore a direct application of A. Meir and J. Moon's Theorem [14]. ☐

Example 2.18. Let $q\geq 2$. Consider *q-uniform partitions*, for which $S=\{q\}$. We have

$$\tau_{\{q\}} = \left(\frac{1}{q-1}\right)^{\frac{1}{q}}, \quad \rho_{\{q\}} = \frac{q-1}{q}\left(\frac{1}{q-1}\right)^{\frac{1}{q}}, \quad \alpha_{\{q\}} = \frac{q}{q-1}, \quad \text{and} \quad \beta_{\{q\}} = \sqrt{\frac{2q^2}{(q-1)^3}}.$$

Therefore, the asymptotic behavior of the number of q-uniform non-crossing partitions with qm vertices is given by

$$[x^{qm}]\,\mathbf{P}_0^{\{q\}}(x,1) \underset{m\to\infty}{=} \sqrt{\frac{q}{2\pi(q-1)^3}}\, m^{-\frac{3}{2}}\left(\frac{q^q}{(q-1)^{q-1}}\right)^m(1+o(1)).$$

Example 2.19. Let $q\geq 1$. Consider *q-multiple partitions*, for which $S=q\mathbb{N}^*$. Since

$$\sum_{n\geq 1}(qn-1)\,x^{qn} = \sum_{n\geq 1} qn\,x^{qn} - \sum_{n\geq 1} x^{qn} = \frac{q\,x^q}{(1-x^q)^2} - \frac{x^q}{1-x^q} = 1 + \frac{(q+1)x^q-1}{(1-x^q)^2},$$

we obtain

$$\tau_{q\mathbb{N}^*} = \left(\frac{1}{q+1}\right)^{\frac{1}{q}}, \quad \rho_{q\mathbb{N}^*} = \frac{q}{q+1}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}, \quad \alpha_{q\mathbb{N}^*} = \frac{q+1}{q}, \quad \text{and} \quad \beta_{q\mathbb{N}^*} = \sqrt{\frac{2\,(q+1)}{q^2}}.$$

Therefore, the asymptotic behavior of the number of q-uniform non-crossing partitions with qm vertices is given by

$$[x^{qm}]\,\mathbf{P}_0^{q\mathbb{N}^*}(x,1) \underset{m\to\infty}{=} \sqrt{\frac{q+1}{2\pi q^3}}\, m^{-\frac{3}{2}} \left(\frac{(q+1)^{q+1}}{q^q}\right)^m (1+o(1)).$$

From the singular behavior of $\mathbf{P}_0^S(x,1)$, and using the composition scheme of Proposition 2.14, we can now extract asymptotic estimates for the number of partitions of $\mathcal{P}^S$ with k crossings.

Proposition 2.20. *Let $k \ge 1$, let S be a non-empty subset of $\mathbb{N}^*$ different from the singleton $\{1\}$, let τ_S, ρ_S, α_S and β_S be the constants described in Definition 2.15, and let $\Phi(k,S)$ denote the maximum value of the potential function*

$$\phi(K) \coloneqq \sum_{i>1} (2i-3)\, n_i(K)$$

over all k-core partitions of $\mathcal{P}^S$. There is a constant Λ_S such that the number of partitions with k crossings, n vertices, and where the size of each block belongs to S is

$$[x^n]\,\mathbf{P}_k^S(x,1) \underset{\substack{n\to\infty \\ \gcd(S)|n}}{=} \Lambda_S\, n^{\frac{\Phi(k,S)}{2}}\, {\rho_S}^{-n}\,(1+o(1)),$$

for n multiple of $\gcd(S)$, while $[x^n]\,\mathbf{P}_k^S(x,1) = 0$ if n is not a multiple of $\gcd(S)$. More precisely, the constant Λ_S can be expressed as

$$\Lambda_S \coloneqq \frac{\gcd(S)\,\Phi(k,S)}{2\,\Gamma\big(\frac{\Phi(k,S)}{2}+1\big)} \sum_K \frac{{\tau_S}^{n_1(K)}}{n(K)} \prod_{i>1} \left(\frac{{\rho_S}^i\,\beta_S\,(2i-5)!!}{2^{i-1}\,(i-1)!}\right)^{n_i(K)},$$

where we sum over the k-core partitions K of $\mathcal{P}^S$ which maximize the potential function $\phi(K)$.

Proof. We exploit the composition scheme obtained in Proposition 2.14 and the description of the singular behavior of $\mathbf{P}_0^S(x,1)$ obtained in Proposition 2.17. In the same lines as the proof of Proposition 2.11, we obtain

$$\mathbf{P}_k^S(x,1) \underset{x\sim\rho_S}{=} \frac{1}{2} \sum_{\substack{K\ k\text{-core} \\ \text{partition of } \mathcal{P}^S}} \frac{\phi(K)\,{\tau_S}^{n_1(K)}}{n(K)} \prod_{i>1} \left(\frac{{\rho_S}^i\,\beta_S\,(2i-5)!!}{2^{i-1}\,(i-1)!}\right)^{n_i(K)} X^{-\phi(K)-2} + O\left(X^{-\phi(K)-1}\right),$$

where $X \coloneqq \sqrt{1-\frac{x}{\rho_S}}$. This expansion is valid in a domain dented at $X = \rho_S$. The asymptotic behavior of this sum is therefore guided by the k-core partitions K of $\mathcal{P}^S$ which maximize the potential $\phi(K)$. Finally, the asymptotic of $[x^n]\,\mathbf{P}_k^S(x,1)$ is obtained combining the contributions of all the singularities $\big\{\rho_S\cdot\xi \;\big|\; \xi\in\mathbb{C},\ \xi^{\gcd(S)}=1\big\}$ of the function $\mathbf{P}_k^S(x,1)$. ☐

Given an arbitrary subset S of $\mathbb{N}^*$, it is in general difficult to describe the k-core partitions of $\mathcal{P}^S$ which maximize the corresponding potential ϕ. The reader is invited to work out examples with uniform partitions (*i.e.* $S=\{q\}$, see Example 2.18) or multiple partitions (*i.e.* $S=q\mathbb{N}^*$, see Example 2.19). Details can be found in [17, Examples 2.33 and 2.35]. We just mention here the case of all partitions with no limitation on the size of the blocks (*i.e.* $S=\mathbb{N}^*$).

Proposition 2.21. *For any $k\ge 1$, the number of partitions with k crossings and n vertices is*

$$[x^n]\,\mathbf{P}_k(x,1) \underset{n\to\infty}{=} \frac{(2k-3)!!}{2^{3k-1}\,k!\,\Gamma\big(k-\frac{1}{2}\big)}\, n^{k-\frac{3}{2}}\, 4^n\,(1+o(1)).$$

3. Chord and hyperchord diagrams

In this section, we consider the family $\mathcal{D}$ of all chord diagrams on the unit circle. Remember that a chord diagram is given by a set of vertices on the unit circle, and a set of chords between them. In particular, we allow isolated vertices, as well as several chord incident to the same vertex, but not multiple chords with the same two endpoints. We let $\mathcal{D}(n,m,k)$ denote the set of chord diagrams in $\mathcal{D}$ with n vertices, m chords, and k crossings. We define the generating functions

$$\mathbf{D}(x,y,z) := \sum_{n,m,k\in\mathbb{N}} |\mathcal{D}(n,m,k)|\, x^n y^m z^k \qquad \text{and} \qquad \mathbf{D}_k(x,y) := [z^k]\, \mathbf{D}(x,y,z),$$

of chord diagrams, and chord diagrams with k crossings, respectively.

Remark 3.1. We insist on the fact that we allow here for isolated vertices in chord diagrams. However, it is essentially equivalent to enumerate chord diagrams or chord configurations (meaning chord diagrams with no isolated vertices). Indeed, their generating functions are related by

$$\mathbf{C}(x,y,z) = \frac{1}{1+x}\,\mathbf{D}\left(\frac{x}{1+x}, y, z\right) \qquad \text{and} \qquad \mathbf{C}_k(x,y) = \frac{1}{1+x}\,\mathbf{D}_k\left(\frac{x}{1+x}, y\right).$$

3.1. **Warming up: crossing-free chord diagrams.** The generating function $\mathbf{D}_0(x,y)$ of crossing-free chord diagrams was studied in [6]. We repeat here their analysis since we will use similar decomposition schemes later for our extension to hyperchord diagrams.

Proposition 3.2 ([6, Equation (22)])**.** *The generating function $\mathbf{D}_0(x,y)$ of crossing-free chord diagrams satisfies the functional equation*

$$y\,\mathbf{D}_0(x,y)^2 + \big(x^2(1+y) - x(1+2y) - 2y\big)\,\mathbf{D}_0(x,y) + x(1+2y) + y = 0. \tag{2}$$

Proof. Consider first a connected crossing-free chord diagram C. By connected we mean here that C is connected as a graph. Call *principal* the chords of C incident to its first vertex (the first after its root). These principal chords split C into smaller crossing-free chord diagrams:

(i) the first (before the first principal chord) and last (after the last principal chord) subdiagrams are both connected chord diagrams,
(ii) each subdiagram inbetween two principal chords consists either in a connected diagram (but not a single vertex), or in two connected diagrams.

This leads to the following functional equation on the generating function $\mathbf{CD}_0(x,y)$ of connected crossing-free chord diagrams:

$$\mathbf{CD}_0(x,y) = x\left(1 + \frac{y\,\mathbf{CD}_0(x,y)^2}{x - y\Big(\mathbf{CD}_0(x,y) - x + \mathbf{CD}_0(x,y)^2\Big)}\right), \tag{3}$$

which can be rewritten as

$$y\,\mathbf{CD}_0(x,y)^3 + y\,\mathbf{CD}_0(x,y)^2 - x(1+2y)\,\mathbf{CD}_0(x,y) + x^2(1+y) = 0. \tag{4}$$

Finally, since a crossing-free chord diagram can be decomposed into connected crossing-free chord diagrams, we have

$$\mathbf{D}_0(x,y) = 1 + \mathbf{CD}_0(x\mathbf{D}_0(x,y)\,,y)\,. \tag{5}$$

Using this equation to eliminate $\mathbf{CD}_0(x,y)$ in Equation (4) leads to the desired formula after straightforward simplifications. □

From the implicit expression of Equation (2), we easily derive the following statement.

Proposition 3.3. *All derivatives $\frac{d^i}{dx^i}\mathbf{D}_0(x,y)$ are rational functions in $\mathbf{D}_0(x,y)$ and x.*

As we are also interested in asymptotic estimates, we proceed to study the singular behavior of $\mathbf{D}_0(x,y)$. As it is proved in [6], the generating function $\mathbf{D}_0(x,y)$ has a unique square-root singularity when y varies around $y=1$:

$$\mathbf{D}_0(x,y) \underset{\substack{y\sim 1\\ x\sim\rho(y)}}{=} d_0(y) - d_1(y)\sqrt{1 - \frac{x}{\rho(y)}} + O\left(1 - \frac{x}{\rho(y)}\right), \tag{6}$$

uniformly with respect to y for y in a small neighborhood of 1, and with $d_0(y)$, $d_1(y)$ and $\rho(y)$ analytic at $y = 1$. In fact, when $y = 1$ we obtain the singular expansion

$$\mathbf{D}_0(x,1) \underset{x\sim\rho(1)}{=} -1 + 3\frac{\sqrt{2}}{2} - \frac{1}{2}\sqrt{-140+99\sqrt{2}}\sqrt{1-\frac{x}{\rho(1)}} + O\left(1-\frac{x}{\rho(1)}\right),$$

with $\rho(1) := \rho = \frac{3}{2} - \sqrt{2} \simeq 0.08578$. This is valid in a domain dented at $x = \rho$. In particular, Equation (6) shows that the singular behavior of $\frac{d^i}{dx^i}\mathbf{D}_0(x,y)$ in a neighborhood of $x = \rho(y)$ is of the form

$$\frac{d^i}{dx^i}\mathbf{D}_0(x,y) \underset{\substack{y\sim 1\\ x\sim\rho(y)}}{=} \frac{d_1(y)\,(2i-3)!!}{\rho(y)^i\,2^i}\left(1-\frac{x}{\rho(y)}\right)^{\frac{1}{2}-i} + O\left(\left(1-\frac{x}{\rho(y)}\right)^{1-i}\right),$$

where we use again the convention that $(-1)!! = 1$ in order to simplify formulas when $i = 1$. This singular expansion will be exploited later in order to get both asymptotic estimates and the limit law for the number of vertices when fixing the number of crossings. Finally, we also need the following values, which appear in [6, Table 5],

$$-\frac{\rho'(1)}{\rho(1)} = \frac{1}{2} + \frac{\sqrt{2}}{2} \qquad \text{and} \qquad -\frac{\rho''(1)}{\rho(1)} - \frac{\rho'(1)}{\rho(1)} + \left(\frac{\rho'(1)}{\rho(1)}\right)^2 = \frac{1}{4} + \frac{\sqrt{2}}{8}. \tag{7}$$

3.2. **Core diagrams.** We now consider chord diagrams with k crossings. As in the previous section, we study them focussing on their cores.

Definition 3.4. *A core diagram is a chord diagram where each chord is involved in a crossing. It is a k-core diagram if it has exactly k crossings. The core $D^\star$ of a chord diagram D is the subdiagram of D formed by all its chords involved in at least one crossing.*

Let K be a core diagram. We let $n(K)$ denote its number of vertices, $m(K)$ denote its number of chords, and $k(K)$ denote its number of crossings. We call *regions* of K the connected components of the complement of K in the unit disk. A region has i *boundary arcs* and j *peaks* if its intersection with the unit circle has i connected arcs and j isolated points. We let $n_{i,j}(K)$ denote the number of regions of K with i boundary arcs and j peaks, and we set $\mathbf{n}(K) := (n_{i,j}(K))_{i,j\in[k]}$. Note that $n(K) = \sum_{i,j} i n_{i,j}(K)$.

Since a crossing only involves two chords, a k-core diagram can have at most $2k$ chords. This immediately implies the following crucial lemma.

Lemma 3.5. *There are only finitely many k-core diagrams.*

Definition 3.6. *We encode the finite list of all possible k-core diagrams K and their parameters $n(K)$, $m(K)$, and $\mathbf{n}(K) := (n_{i,j}(K))_{i,j\in[k]}$ in the k-core diagram polynomial*

$$\mathbf{KD}_k(\mathbf{x},y) := \mathbf{KD}_k(x_{i,j},y) := \sum_{\substack{K\ k\text{-core}\\ \text{diagram}}} \frac{\mathbf{x}^{\mathbf{n}(K)}y^{m(K)}}{n(K)} := \sum_{\substack{K\ k\text{-core}\\ \text{diagram}}} \frac{1}{n(K)} \prod_{i,j\geq 0} {x_{i,j}}^{n_{i,j}(K)}\, y^{m(K)}.$$

3.3. **Generating function of chord diagrams with k crossings.** In this section, we express the generating function $\mathbf{D}_k(x,y)$ of chord diagrams with k crossings as a rational function of the generating function $\mathbf{D}_0(x,y)$ of crossing-free diagrams, using the k-core diagram polynomial $\mathbf{KD}_k(\mathbf{x},y)$ defined in the previous section.

First, we say that a chord diagram D is *weakly rooted* if we have marked an arc between two consecutive vertices of its core $K^\star$. Again, we have the following rerooting lemma.

Lemma 3.7. *For any core diagram K, the number $D_K(n,m)$ of rooted chord diagrams with n vertices, m chords, and core K and the number $\bar{D}_K(n,m)$ of weakly rooted chord diagrams with n vertices, m chords, and core K are related by $n(K)D_K(n,m) = n\bar{D}_K(n,m)$.*

As for matchings, we can now construct any chord diagram with k crossings by inserting crossing-free subdiagrams in the regions left by its k-core. We can therefore derive the following expression for the generating function $\mathbf{D}_k(x,y)$ of diagrams with k crossings, in terms of the generating function $\mathbf{D}_0(x,y)$ of crossing-free diagrams, of the k-core diagram polynomial $\mathbf{KD}_k(x,y)$, and of the polynomials

$$\mathbf{D}_0^n(y) := [x^n]\,\mathbf{D}_0(x,y) \qquad \text{and} \qquad \mathbf{D}_0^{\le p}(x,y) := \sum_{n\le p} \mathbf{D}_0^n(y)\,x^n = \sum_{\substack{n\le p\\ m\ge 0}} |\mathcal{D}(n,m,0)|\,x^n y^m.$$

Proposition 3.8. *For any $k \ge 1$, the generating function $\mathbf{D}_k(x,y)$ of chord diagrams with k crossings is given by*

$$\mathbf{D}_k(x,y) = x\frac{d}{dx}\mathbf{KD}_k\left(x_{0,j} \leftarrow \frac{\mathbf{D}_0^j(y)}{x^j},\ x_{i,j} \leftarrow \frac{x^i}{(i-1)!}\frac{d^{i-1}}{dx^{i-1}}\frac{\mathbf{D}_0(x,y)-\mathbf{D}_0^{\le i+j}(x,y)}{x^{i+j+1}},\ y\right).$$

In particular, $\mathbf{D}_k(x,y)$ is a rational function of $\mathbf{D}_0(x,y)$ and x.

Proof. Consider a rooted crossing-free chord diagram D, whose vertices are labeled from 1 to n clockwise starting from the root. Let $\mathbf{j} := (j_1,\dots,j_i)$ be a list of i positive integers whose sum is j. We say that D is $\mathbf{j}$-*marked* if we have marked i vertices of D, including the first vertex labeled 1, in such a way that there is at least j_k+1 vertices between the k^{th} and $(k+1)^{\text{th}}$ marked vertices, for any $k \in [i]$. More precisely, if we mark the vertices labeled by $1 = \alpha_1 < \dots < \alpha_i$ and set by convention $\alpha_{i+1} = n+1$, then we require that $\alpha_{k+1} - \alpha_k > j_k + 1$ for any $k \in [i]$. Note that if D has less than $2i+j$ vertices, then it cannot be $\mathbf{j}$-marked. Otherwise, if D has at least $2i+j$ vertices, we have $\binom{n-i-j-1}{i-1}$ ways to place these i marks. Therefore, the generating function of the rooted $\mathbf{j}$-marked crossing-free chord diagrams is given by

$$\frac{x^{2i+j}}{(i-1)!}\frac{d^{i-1}}{dx^{i-1}}\frac{\mathbf{D}_0(x,y)-\mathbf{D}_0^{\le i+j}(x,y)}{x^{i+j+1}}.$$

Consider now a weakly rooted chord diagram D with k crossings. We decompose this diagram into several subdiagrams as follows. On the one hand, the core $D^\star$ contains all crossings of D. This core is rooted by the root of D. On the other hand, each region R of $D^\star$ contains a crossing-free subdiagram D_R. We root this subdiagram D_R as follows:

(i) if the root of D points out of R, then D_R is just rooted by the root of D;
(ii) otherwise, D_R is rooted on the first boundary arc of $D^\star$ before the root of D in clockwise direction.

Moreover, we mark the first vertex of each boundary arc of R. Note that we do not mark the peaks. Thus, if the region R has i boundary arcs, and if the k^{th} and $(k+1)^{\text{th}}$ boundary arcs of R are separated by j_k peaks, then we obtain in this region R of $D^\star$ a rooted $(j_1,\dots,j_i)$-marked crossing-free subdiagram D_R. Observe that their is a difference of behavior between

(i) the regions R with no boundary arcs and only j peaks, which are filled in by a crossing-free chord diagram D_R on precisely j vertices, and
(ii) the regions R with at least one boundary arc, whose corresponding chord diagram D_R can have arbitrarily many additional vertices.

Reciprocally we can reconstruct the chord diagram D from its rooted core $D^\star$ and its rooted and marked crossing-free subdiagrams D_R. We thus obtain that the generating function $\mathbf{D}_k(x,y)$ from the k-core diagram polynomial $\mathbf{KD}_k(\mathbf{x},y)$ by replacing a region with $i \ne 0$ boundary arcs and j peaks by

$$\frac{x^{i}}{(i-1)!}\frac{d^{i-1}}{dx^{i-1}}\frac{\mathbf{D}_0(x,y)-\mathbf{D}_0^{\le i+j}(x,y)}{x^{i+j+1}},$$

and a region with no boundary arcs but j peaks by $\mathbf{D}_0^j(y)\,/x^j$. This is precisely the formula stated in the proposition.

The rationality of this function thus follows from Proposition 3.3, since $\mathbf{D}_0^j(x,y)$ and $\mathbf{D}_0^{\le j}(x,y)$ are both polynomials in x and y, and y can be eliminated as in the proof of Proposition 3.3. $\square$

3.4. **Asymptotic analysis.** Similarly to our asymptotic analysis in Section 2.4, we can obtain asymptotic results for the number of chord diagrams with k crossings.

Proposition 3.9. *For any $k \geq 1$, the number of chord diagrams with k crossings and n vertices is*

$$[x^n]\,\mathbf{D}_k(x,1) \underset{n\to\infty}{=} \frac{d_0(1)^{3k}\,d_1(1)\,(2k-3)!!}{(2\rho)^{k-1}\,k!\,\Gamma(k-\frac{1}{2})}\,n^{k-\frac{3}{2}}\,\rho^{-n}\,(1+o(1)),$$

where

$$\rho^{-1} = 6+4\sqrt{2}, \qquad d_0(1) = -1+3\frac{\sqrt{2}}{2}, \qquad \textit{and} \qquad d_1(1) = \frac{1}{2}\sqrt{-140+99\sqrt{2}}.$$

Proof. We apply singularity analysis on the composition scheme given by Proposition 3.8. In our analysis, it is more convenient to express the k-core diagram polynomial $\mathbf{KD}_k(\mathbf{x},1)$ as

$$\mathbf{KD}_k(\mathbf{x},1) = \sum_{\substack{K\ k\text{-core}\\ \text{diagram}}} \frac{1}{n(K)} \prod_{j\geq 0} {x_{0,j}}^{n_{0,j}(K)} \prod_{\substack{i\geq 1\\ j\geq 0}} {x_{i,j}}^{n_{i,j}(K)}.$$

The resulting expression for $\mathbf{D}_k(x,1)$ is

$$x\frac{d}{dx}\sum_{\substack{K\ k\text{-core}\\ \text{diagram}}} \frac{1}{n(K)} \prod_{j\geq 0}\left(\frac{\mathbf{D}_0^j(1)}{x^j}\right)^{n_{0,j}(K)} \prod_{\substack{i\geq 1\\ j\geq 0}}\left(\frac{x^i}{(i-1)!}\frac{d^{i-1}}{dx^{i-1}}\frac{\mathbf{D}_0(x,1)-\mathbf{D}_0^{\leq i+j}(x,1)}{x^{i+j+1}}\right)^{n_{i,j}(K)}.$$

Analyzing this function around $x=\rho$ boils down to analyzing the generating function

$$\rho\frac{d}{dx}\sum_{\substack{K\ k\text{-core}\\ \text{diagram}}} \frac{1}{n(K)} \prod_{j\geq 0}\left(\frac{\mathbf{D}_0^j(1)}{\rho^j}\right)^{n_{0,j}(K)} \prod_{\substack{i\geq 1\\ j\geq 0}}\left(\frac{1}{\rho^{j+1}\,(i-1)!}\frac{d^{i-1}}{dx^{i-1}}\mathbf{D}_0(x,1)\right)^{n_{i,j}(K)}.$$

Observe that we forget the terms of the form $\mathbf{D}_0^{\leq i+j}(x,1)$ as they are polynomials in x, and thus analytic functions around $x=\rho$. In order to simplify the expressions, we set

$$\xi(K) := \frac{1}{n(K)} \prod_{j\geq 0}\left(\frac{\mathbf{D}_0^j(1)}{\rho^j}\right)^{n_{0,j}(K)} \prod_{\substack{i\geq 1\\ j\geq 0}}\left(\frac{1}{\rho^{j+1}\,(i-1)!}\right)^{n_{i,j}(K)}.$$

Let $X := \sqrt{1-\frac{x}{\rho}}$. Developing $\mathbf{D}_0(x,1)$ using its Puiseux's expansion (6) around $x=\rho$ we obtain

$$\begin{aligned}
\mathbf{D}_k(x,1) &\underset{x\sim\rho}{=} \rho\frac{d}{dx}\sum_{\substack{K\ k\text{-core}\\ \text{diagram}}} \xi(K)\prod_{\substack{i\geq 1\\ j\geq 0}}\left(\frac{d^{i-1}}{dx^{i-1}}\mathbf{D}_0(x,1)\right)^{n_{i,j}(K)}\\
&\underset{x\sim\rho}{=} \rho\frac{d}{dx}\sum_{\substack{K\ k\text{-core}\\ \text{diagram}}} \xi(K)\prod_{\substack{i\geq 1\\ j\geq 0}}\left(\frac{d^{i-1}}{dx^{i-1}}\big(d_0(1)+d_1(1)\,X\,+O\left(X^2\right)\big)\right)^{n_{i,j}(K)}\\
&\underset{x\sim\rho}{=} \rho\frac{d}{dx}\sum_{\substack{K\ k\text{-core}\\ \text{diagram}}} \xi(K)\prod_{j\geq 0} d_0(1)^{n_{1,j}(K)}\prod_{\substack{i>1\\ j\geq 0}}\left(\frac{d_1(1)\,(2i-5)!!}{2^{i-1}\,\rho^{i-1}}\,X^{3-2i}\,+O\left(X^{4-2i}\right)\right)^{n_{i,j}(K)}\\
&\underset{x\sim\rho}{=} \rho\frac{d}{dx}\sum_{\substack{K\ k\text{-core}\\ \text{diagram}}} \zeta(K)\,X^{-\psi(K)}\,+O\left(X^{-\psi(K)+1}\right)\\
&\underset{x\sim\rho}{=} \rho\sum_{\substack{K\ k\text{-core}\\ \text{diagram}}} -\zeta(K)\,\psi(K)\,X^{-\psi(K)-2}\,+O\left(X^{-\psi(K)-1}\right),
\end{aligned}$$

where

$$\zeta(K) := \frac{1}{n(K)} \prod_{j\geq 0} \left(\frac{\mathbf{D}_0^j(1)}{\rho^j}\right)^{n_{0,j}(K)} d_0(1)^{n_{1,j}(K)} \prod_{\substack{i>1\\ j\geq 0}} \left(\frac{d_1(1)\,(2i-5)!!}{2^{i-1}\,\rho^{i+j}\,(i-1)!}\right)^{n_{i,j}(K)},$$

$$\text{and} \qquad \psi(K) := \sum_{\substack{i>1\\ j\geq 0}} (2i-3)\, n_{i,j}(K).$$

Following the same lines as in Section 2.4, the main contribution to the asymptotic arise from the k-core diagrams which maximizes $\psi(K)$. These k-core diagrams satisfy $n_{k,0}(K) = 1$, $n_{1,0}(K) = 3k$ and $n_{i,j}(K) = 0$ for all $(i,j) \neq (k,0), (1,0)$. Consequently $\psi(K) = 2k - 3$. Therefore,

$$\mathbf{D}_k(x,1) \underset{x\sim\rho}{=} \frac{d_0(1)^{3k}\, d_1(1)\,(2k-3)!!}{(2\rho)^{k-1}\, k!}\, X^{1-2k} + O\left(X^{2-2k}\right),$$

and we conclude applying the Transfer Theorem for singularity analysis [8, 9]. ☐

Finally, with the same techniques, we can also compute the limiting distribution of the number of edges in a k-chord diagram with n vertices, chosen uniformly at random.

Theorem 3.10. *The number of edges in a chord diagram with k crossings and n vertices, chosen uniformly at random, follows a normal distribution with expectation μ_n and variance σ_n, where*

$$\mu_n = \left(\frac{1}{2} + \frac{\sqrt{2}}{2}\right) n\,(1+o(1)) \qquad \textit{and} \qquad \sigma_n = \left(\frac{1}{4} + \frac{\sqrt{2}}{8}\right) n\,(1+o(1)).$$

Proof. Direct application of the Quasi-Powers Theorem [11], by means of the values computed in Equation (7). The main contribution on the analysis arises from maximal k-core diagrams. Observe that the constants defining the expectation and the variance are exactly the same as in the planar configurations. ☐

3.5. **Random generation.** In this section, we provide random generators for the combinatorial family of chord diagrams with a given number of crossings, using the methodology of Bolzmann samplers. We proceed in three steps, obtaining random generators for:

(i) connected crossing-free chord diagrams,
(ii) all crossing-free chord diagrams,
(iii) chord diagrams with precisely k crossings.

Once we have a Boltzmann sampler for crossing-free chord diagrams, the design of a random generator for chord diagrams with precisely k crossings follows exactly the same lines as in Section 2.5. In this section, we therefore only discuss Steps (i) and (ii) above.

We first describe a Bolzmann sampler for connected crossing-free chord diagrams. It is convenient to write Equation (3) (with $y = 1$) in the form

$$\mathbf{CD}_0(x,1) = x\left(1 + \frac{\mathbf{CD}_0(x,1)^2}{x} \sum_{r=0}^{\infty} \left(\frac{\mathbf{CD}_0(x,1) - x + \mathbf{CD}_0(x,1)^2}{x}\right)^r\right).$$

The smallest singularity of $\mathbf{CD}_0(x,1)$ is located at $\rho_0 \simeq 0.09623$. For $r \geq 0$, write

$$\mathbf{CD}_0^r(x) = \frac{\mathbf{CD}_0(x,1)^2}{x}\left(\frac{\mathbf{CD}_0(x,1) - x + \mathbf{CD}_0(x,1)^2}{x}\right)^r$$

and fix $\theta \in (0, \rho_0)$. Observe that the combinatorial class associated to $\mathbf{CD}_0^r(x)$ can be defined by means of cartesian products and unions of connected crossing-free chord diagrams, hence the Boltzmann sampler $\Gamma\mathbf{CD}_0^r(\theta)$ is immediately defined from $\Gamma\mathbf{CD}_0(\theta)$. Let

$$p_r(\theta) := \frac{\mathbf{CD}_0^r(\theta)}{\mathbf{CD}_0(\theta,1)} \qquad \text{and} \qquad p_{-1}(\theta) := \frac{\theta}{\mathbf{CD}_0(\theta,1)}.$$

Then $P(\theta) := \{p_r(\theta)\}_{r\geq -1}$ defines a discrete probability distribution. Now we can define the Boltzmann sampler $\Gamma\mathbf{CD}_0(\theta)$ by

$$\Gamma\mathbf{CD}_0(\theta) := P(\theta) \longrightarrow \Gamma\mathbf{CD}_0^r(\theta)\,.$$

As it happened in the perfect matching situation, the branching process defined with this Boltzmann sampler is subcritical, hence the algorithm finishes in expected finite time.

We now describe a random sampler for general crossing-free chord diagrams. For this, we analyze Equation (5), which describes the counting formula for general chord diagrams by means of a composition scheme with the generating function associated to connected chord diagrams. The Boltzmann sampler in this situation is reminiscent to the L-substitution that appears in [10]. Fix $\theta' \in (0, \frac{3}{2} - \sqrt{2})$ (recall that the smallest singularity of $\mathbf{D}_0(\theta)$ is located at $\rho = \frac{3}{2} - \sqrt{2}$), and define

$$q_s(\theta') := \theta'^s \mathbf{D}_0(\theta')^{s-1}\,[x^s]\,\mathbf{CD}_0^s(x) \qquad \text{and} \qquad p_{-1}(\theta) := \frac{1}{\mathbf{D}_0(\theta')}.$$

Then $Q(\theta') := \{q_s(\theta')\}_{s\geq -1}$ defines a discrete probability distribution and we can apply the same argument as in the case of connected objects. Once more, the choice of a parameter smaller than the smallest singularity ensures that the algorithm finishes with an expected finite time. Observe that in the second Boltzmann sampler, a choice of a connected chord diagram is needed. This is performed using a rejection process over the Boltzmann sampler for connected chord diagrams.

3.6. **Extension to hyperchord diagrams.** As from matchings to partitions, we can extend the results of this section from chord diagrams to hyperchord diagrams. We start here with al hyperchord diagrams and extend our results to hyperchord diagrams with restricted block sizes later in Section 3.7. A *hyperchord* is the convex hull of finitely many points of the unit circle. Given a point set V on the circle, a *hyperchord diagram* on V is a set of hyperchords with vertices in V. Note that we allow isolated vertices in hyperchord diagrams. As for partitions, a *crossing* between two hyperchords U, V is a pair of crossing chords u_1u_2 and v_1v_2, with $u_1, u_2 \in U$ and $v_1, v_2 \in V$. We consider the family $\mathcal{H}$ of hyperchord diagrams, and we let $\mathcal{H}(n, m, k)$ be the set of hyperchord diagrams with n vertices, m hyperchords, and k crossings, counted with multiplicities. We set

$$\mathbf{H}(x, y, z) := \sum_{n,m,k} |\mathcal{H}(n, m, k)|\, x^n y^m z^k \qquad \text{and} \qquad \mathbf{H}_k(x, y) := [z^k]\,\mathbf{H}(x, y, z).$$

As for chord diagrams, our first step is to study the generating function $\mathbf{H}_0(x, y)$ of crossing-free hyperchord diagrams. We extend here the analysis of P. Flajolet and M. Noy for chord diagrams [6] that we presented in Section 3.1. Note that two non-crossing hyperchords U, V can share at most two vertices. Moreover, if U and V share two vertices, then they lie on opposite sides of the chord joining them, and we say that U, V are *kissing* hyperchords.

Proposition 3.11. *The generating function $\mathbf{H}_0(x, y)$ of crossing-free hyperchord diagrams satisfies the functional equation*

$$p_3(x,y)\,\mathbf{H}_0(x,y)^3 + p_2(x,y)\,\mathbf{H}_0(x,y)^2 + p_1(x,y)\,\mathbf{H}_0(x,y) + p_0(x,y) = 0, \tag{8}$$

where

$$\begin{aligned}
p_0(x,y) &:= -2\,x^2 - x + 2\,x\,y^3 + y^2 + x^2\,y^4 - 7\,x^2\,y - 7\,x^2\,y^2 - x^2\,y^3 - 3\,x\,y,\\
p_1(x,y) &:= -2\,x^3 - 2\,x^3\,y^4 - 8\,x^3\,y + 2\,x - 3\,y^2 - 12\,x^3\,y^2 - 8\,x^3\,y^3\\
&\qquad + 6\,x\,y - x^2\,y^4 + x^2 + 4\,x^2\,y + 4\,x^2\,y^2 - 4\,x\,y^3,\\
p_2(x,y) &:= x^2\,y^3 + x^2 + 3\,x^2\,y^2 - x - 3\,x\,y + 2\,x\,y^3 + 3\,x^2\,y + 3\,y^2,\\
p_3(x,y) &:= -y^2.
\end{aligned}$$

Proof. The proof is similar to that of Proposition 3.2. We first consider connected crossing-free hyperchord diagrams, with generating function $\mathbf{CH}_0(x, y)$. We decompose them according to their *principal* hyperchords (those incident to the first vertex, and with at least two vertices). If a connected hyperchord diagram is neither an isolated vertex, nor a single hyperchord with a single vertex, then it has at least one principal hyperchord. Its principal hyperchords can then be grouped into clusters such that

(i) the principal hyperchords in a given cluster form a sequence of kissing hyperchords, and
(ii) principal hyperchords of distinct clusters share only the first vertex of the diagram.

Note that each cluster is either a single chord, or can be considered as a sequence of r kissing principal hyperchords of size at least 3, whose $r+1$ principal boundary chords may or not be principal hyperchords of the diagram. It remains to fill in the gaps left by the principal hyperchords in the hyperchord diagram:

(i) the first (before the first cluster), and the last (after the last cluster) gaps contain connected crossing-free hyperchord diagrams,
(ii) each gap between two consecutive clusters, as well as each gap between two consecutive vertices of a principal hyperchord, contains either a connected crossing-free hyperchord diagram with at least two vertices, or two disconnected crossing-free hyperchord diagrams.

This decomposition directly translates to the functional equation

$$\mathbf{CH}_0(x,y) = x\,(1+y) + \frac{g\,\mathbf{CH}_0(x,y)^2}{x\,(1+y)\,(1-f\,g)},$$

where

$$f := \frac{\mathbf{CH}_0(x,y)^2 + \mathbf{CH}_0(x,y) - x\,(1+y)}{x^2\,(1+y)^2} \quad \text{and} \quad g := x\,y\,(1+y) + \frac{x^2\,y\,(1+y)^4\,f}{1 - x\,(1+y)\,f - x\,y\,(1+y)^2\,f}.$$

Finally, since a crossing-free hyperchord diagram can be decomposed into connected crossing-free hyperchord diagrams, we have

$$\mathbf{H}_0(x,y) = 1 + \mathbf{CH}_0(x\mathbf{H}_0(x,y), y)\,. \tag{9}$$

Eliminating $\mathbf{CH}_0(x,y)$ from these equations leads to the desired formula after simplifications. □

Remark 3.12. If we forget the variable y which encodes the number of hyperchords, and if we forbid isolated vertices in hyperchord diagrams, the resulting generating function

$$\widetilde{\mathbf{H}}_0(x) := \frac{1}{1+x}\,\mathbf{H}_0\left(\frac{x}{1+x}, 1\right)$$

satisfies the functional equation

$$(1+x)^5\,\widetilde{\mathbf{H}}_0(x)^3 - (1+x)^2(9x^2+4x+3)\,\widetilde{\mathbf{H}}_0(x)^2 + (23x^3-7x^2+5x+3)\,\widetilde{\mathbf{H}}_0(x) + (17x^2-1) = 0.$$

It was already obtained by M. Klazar in [13] with a slightly different decomposition scheme.

The next proposition studies the asymptotic behaviour of $\mathbf{H}_0(x,y)$ around $y=1$. Observe that we cannot apply A. Meir and J. Moon's smooth implicit-function Theorem [14] since the coefficients in Equation (8) are not all positive. We can use instead the fact that $\mathbf{H}_0(x,y)$ is an algebraic function. Another technique, by means of more elaborated arguments, will be presented in the context of Subsection 3.7, which covers this proposition.

Proposition 3.13. *The smallest singularity of the generating function $\mathbf{H}_0(x,1)$ of crossing-free hyperchord diagrams is located at the smallest real root $\rho \simeq 0.015391$ of the polynomial*

$$R(x) := 256\,x^4 - 768\,x^3 + 736\,x^2 - 336\,x + 5,$$

Moreover, when y varies uniformly in a small neighborhood of 1, the singular expansion of $\mathbf{H}_0(x,y)$ is

$$\mathbf{H}_0(x,y) \underset{y\sim 1}{=} h_0(y) - h_1(y)\sqrt{1-\frac{x}{\rho(y)}} + O\left(1+\frac{x}{\rho(y)}\right),$$

valid in a domain dented at $\rho(y)$ (for each choice of y), where $h_0(y)$, $h_1(y)$ and $\rho(y)$ are analytic functions around $y=1$, with

$$\rho(1) = \rho \simeq 0.015391, \qquad h_0(1) \simeq 1.034518 \qquad \textit{and} \qquad h_1(1) \simeq 0.00365515.$$

Proof. We use the methodology of [9, Section VII.7]. We write Equation (8) in the form $P(x, y, \mathbf{H}_0(x, y)) = 0$, where $P(x, y, z)$ is a polynomial with integer coefficients.

We first study the problem when $y = 1$. It is clear that $\mathbf{H}_0(x, 1)$ is analytic at $x = 0$. Consequently, according to [9, Lemma VII.4], $\mathbf{H}_0(x, 1)$ can be analytically continued along any simple path emanating from the origin that does not cross any point of the set in which both $P(x, 1, z)$ and $\frac{d}{dz}P(x, 1, z)$ vanish. This set is discrete and, by means of Elimination Theory for algebraic functions (in this case, eliminating variable z), can be written as the set of roots of

$$R(x) := 256\,x^4 - 768\,x^3 + 736\,x^2 - 336\,x + 5.$$

By Pringsheim's Theorem the dominant singularity of $\mathbf{H}_0(x, 1)$ (if exists) is a real positive number. Additionally, $R(x)$ has two real roots, one which is smaller than 1 and another with is greater than 1. Hence, we conclude that the smallest singularity of $\mathbf{H}_0(x, 1)$ is the smallest root $\rho \simeq 0.015391$ of the polynomial $R(x)$. In particular $\mathbf{H}_0(\rho, 1)$ satisfies the equation $P(\rho, 1, \mathbf{H}_0(\rho, 1)) = 0$, and is approximately equal to $\mathbf{H}_0(\rho, 1) = h_0(1) \simeq 1.034518$.

We now proceed to study the nature of $\mathbf{H}_0(x, 1)$ around $x = \rho$. As $\mathbf{H}_0(x, 1)$ is algebraic, we can develop it around its smallest singularity using its *Puiseux expansion*, and exploiting the so-called *Newton Polygon Method*. See [9, Page 498]. With this purpose, write $U := 1 - \frac{x}{\rho}$, and $\mathbf{H}_0(x, 1) = \mathbf{H}_0(\rho, 1) + c\,U^{\alpha}\,(1 + o(1))$. By means of indeterminate coefficients we find the correct value of α: developing the relation $P(\rho\,(1 - U^{\frac{1}{\alpha}}), 1, \mathbf{H}_0(\rho, 1) + c\,U^{\alpha}) = 0$ we obtain that $\alpha = \frac{1}{2}$. Once we know this, by indeterminate coefficients on the expression $P(\rho, 1, \mathbf{H}_0(\rho, 1)) = 0$ we obtain that $h_1(1) \simeq 0.00365515$.

Finally, we continue analyzing $\mathbf{H}_0(x, y)$ when y moves in a small neighbourhood around $y = 1$. Using the same arguments, for a fixed value of y close to 1, the smallest singularity $\rho(y)$ of $\mathbf{H}_0(x, y)$ satisfies that $R(\rho(y), y) = 0$, with

$$\begin{aligned} R(x, y) = \quad & (1 + 4\,y + 10\,y^2 + 4\,y^3 + y^4) \\ & + (-6 - 40\,y - 142\,y^2 - 304\,y^3 - 390\,y^4 - 296\,y^5 - 130\,y^6 - 32\,y^7 - 4\,y^8)\,x \\ & + (13 + 100\,y + 360\,y^2 + 748\,y^3 + 922\,y^4 + 636\,y^5 + 192\,y^6 - 12\,y^7 - 15\,y^8)\,x^2 \\ & + (-12 - 96\,y - 336\,y^2 - 672\,y^3 - 840\,y^4 - 672\,y^5 - 336\,y^6 - 96\,y^7 - 12\,y^8)\,x^3 \\ & + (4 + 32\,y + 112\,y^2 + 224\,y^3 + 280\,y^4 + 224\,y^5 + 112\,y^6 + 32\,y^7 + 4\,y^8)\,x^4. \end{aligned}$$

As the coefficients (which depend on y) of $R(x, y)$ do not vanish at $y = 1$, we conclude that $\rho(y)$ is an analytic function in a neighbourhood of $y = 1$, and that the singularity type is invariably of square root type. □

The analysis carried out in Proposition 3.13 can be exploited in order to obtain the limit distribution for the number of hyperchords in a crossing-free hyperchord diagram of prescribed size uniformly choosen at random. For each y in a neighbourhood of 1 the singular expansion of $\mathbf{H}_0(x, y)$ is of square-root type, hence by the Quasi-Powers Theorem [11], the limiting law is normally distributed. We can compute the expectation and the variance from polynomial $R(x, y)$ in Proposition 3.13: as $R(\rho(y), y) = 0$ by iterated derivations with respect to y we obtain closed formulas of both $\rho'(y)$ and $\rho''(y)$ in terms of y, $\rho(y)$ (and $\rho'(y)$ in the case of $\rho''(y)$). Writing $y = 1$ we get approximate values: these computations give $\rho'(1) \simeq -0.031243$ and $\rho''(1) \simeq 0.080456$, hence the expectation and the variance for this limiting distribution are $\mu\,n\,(1 + o(1))$ and $\sigma^2\,n\,(1 + o(1))$, where

$$\mu = -\frac{\rho'(1)}{\rho(1)} \simeq 2.029890 \qquad \text{and} \qquad \sigma^2 = -\frac{\rho''(1)}{\rho(1)} - \frac{\rho'(1)}{\rho(1)} + \left(\frac{\rho'(1)}{\rho(1)}\right)^2 \simeq 0.923054.$$

Now that we have obtained the asymptotic behavior of crossing-free hyperchord diagrams, we can proceed to the study of hyperchord diagrams with k crossings. Using a similar method as in Section 2.2, we obtain the following expression of the generating function $\mathbf{H}_k(x, y)$ of hyperchord diagrams with k crossings, in terms of the k-core hyperchord diagram polynomial

$$\mathbf{KH}_k(\mathbf{x}, y) := \mathbf{KH}_k(x_{i,j}, y) := \sum_{\substack{K \text{ } k\text{-core} \\ \text{hyperchord} \\ \text{diagram}}} \frac{\mathbf{x}^{\mathbf{n}(K)} y^{m(K)}}{n(K)} := \sum_{\substack{K \text{ } k\text{-core} \\ \text{hyperchord} \\ \text{diagram}}} \frac{1}{n(K)} \prod_{i,j \geq 0} {x_{i,j}}^{n_{i,j}(K)} \, y^{m(K)},$$

and of the polynomials

$$\mathbf{H}_0^n(y) := [x^n]\, \mathbf{H}_0(x, y) \qquad \text{and} \qquad \mathbf{H}_0^{\leq p}(x, y) := \sum_{n \leq p} \mathbf{H}_0^n(y)\, x^n = \sum_{\substack{n \leq p \\ m \geq 0}} |\mathcal{H}(n, m, 0)|\, x^n y^m.$$

Proposition 3.14. *For any $k \geq 1$, the generating function $\mathbf{H}_k(x, y)$ of the hyperchord diagrams with k crossings is given by*

$$\mathbf{H}_k(x, y) = x \frac{d}{dx} \mathbf{KH}_k\left(x_{0,j} \leftarrow \frac{\mathbf{H}_0^j(y)}{x^j},\, x_{i,j} \leftarrow \frac{x^i}{(i-1)!} \frac{d^{i-1}}{dx^{i-1}} \frac{\mathbf{H}_0(x, y) - \mathbf{H}_0^{\leq i+j}(x, y)}{x^{i+j+1}},\, y \right).$$

In particular, $\mathbf{H}_k(x, y)$ is a rational function of the generating function $\mathbf{H}_0(x, y)$ and of the variables x and y.

Note that, contrarily to the cases of matchings, partitions and diagrams, we cannot anymore eliminate y in the expression of Proposition 3.11.

Finally, using the expression of the generating function $\mathbf{H}_k(x, y)$ given by Proposition 3.14, we can derive the asymptotic behavior of the number of hyperchord diagrams with k crossings. The analysis is identical to that of the proof of Proposition 3.9.

Proposition 3.15. *For any $k \geq 1$, the number of hyperchord diagrams with k crossings and n vertices is*

$$[x^n]\, \mathbf{D}_k(x, 1) \underset{n \to \infty}{=} \frac{h_0(1)^{3k}\, h_1(1)\, (2k-3)!!}{(2\rho)^{k-1}\, k!\, \Gamma(k - \frac{1}{2})}\, n^{k - \frac{3}{2}}\, \rho^{-n}\, (1 + o(1)),$$

where $\rho \simeq 0.015391$ is the smallest real root of $R(x) := 256\,x^4 - 768\,x^3 + 736\,x^2 - 336\,x + 5$, and where $h_0(1) \simeq 1.034518$ and $h_1(1) \simeq 0.00365515$ (see also Proposition 3.13).

3.7. **Extension to hyperchord diagrams with restricted block sizes.** As for partition, we conclude this section with hyperchord diagrams where we restrict the sizes of the hyperchords. For a non-empty subset S of $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$, we denote by $\mathcal{H}^S$ the family of hyperchord diagrams, where the cardinality of each hyperchord belongs to S. For example, chord diagrams are hyperchord diagrams where each hyperchord has size 2, *i.e.* $\mathcal{D} = \mathcal{H}^{\{2\}}$. We consider here the generating function $\mathbf{H}_k^S(x, y)$ of hyperchord diagrams of $\mathcal{H}^S$ with k crossings.

Once again, our first step is to compute the generating function $\mathbf{H}_0^S(x, y)$ of crossing-free hyperchord diagrams of $\mathcal{H}^S$. Adapting the decomposition scheme described in the proof of Proposition 3.11, we obtain the following statement.

Proposition 3.16. *The generating function $\mathbf{CH}_0^S(x, y)$ of connected crossing-free hyperchord diagrams of $\mathcal{H}^S$ satisfies the functional equation:*

$$\mathbf{CH}_0^S(x, y) = x\,(1 + \delta_1\, y) + \frac{g\, \mathbf{CH}_0^S(x, y)^2}{x\,(1 + \delta_1\, y)\,(1 - f\, g)},$$

where

$$f := \frac{\mathbf{CH}_0^S(x, y)^2 + \mathbf{CH}_0^S(x, y) - x\,(1 + \delta_1\, y)}{x^2\,(1 + \delta_1\, y)^2}$$

$$g := \delta_2\, x\, y\, (1 + \delta_1\, y) + \frac{x\,(1 + \delta_1\, y)\,(1 + \delta_2\, y)\, h}{1 - h},$$

and $$h := y\,(1 + \delta_2\, y) \sum_{s \in S \setminus \{1,2\}} \big(x\,(1 + \delta_1\, y)\, f\big)^{s-2},$$

and where $\delta_1 = 0$ if $1 \notin S$ and $\delta_1 = 1$ otherwise, and similarly $\delta_2 = 0$ if $2 \notin S$ and $\delta_2 = 1$ otherwise In turn, the generating function $\mathbf{H}_0^S(x,y)$ of crossing-free hyperchord diagrams of $\mathcal{H}^S$ can be expressed from $\mathbf{CH}_0^S(x,y)$ as

$$\mathbf{H}_0^S(x,y) = 1 + \mathbf{CH}_0^S\big(x\mathbf{H}_0^S(x,y), y\big)\,. \tag{10}$$

Although we cannot find a nice closed formula for $\mathbf{H}_0^S(x,y)$ in general, the situation is simpler for the following examples.

Example 3.17. Let $q \geq 3$. Consider *q-uniform hyperchord diagrams*, for which $S = \{q\}$. The generating function of connected crossing-free q-uniform hyperchord diagrams satisfies the functional equation

$$\left(\mathbf{CH}_0^{\{q\}}(x,y) - x\right) x^{q-1} - y\,\mathbf{CH}_0^{\{q\}}(x,y)\,\left(\mathbf{CH}_0^{\{q\}}(x,y)^2 + \mathbf{CH}_0^{\{q\}}(x,y) - x\right)^{q-1} = 0$$

and therefore we get

$$\left(\mathbf{H}_0^{\{q\}}(x,y) - 1 - x\,\mathbf{H}_0^{\{q\}}(x,y)\,\right) x^{q-1} - y\left(\mathbf{H}_0^{\{q\}}(x,y) - 1\right)\left(\mathbf{H}_0^{\{q\}}(x,y) - 1 - x\right)^{q-1} = 0$$

Rephrasing the arguments of Proposition 3.13 we can conclude that $\mathbf{CH}_0^{\{q\}}(x,1)$ has a unique smallest real singularity and its singular behaviour is of square-root type in a domain dented at its singular point.

Finally, we observe that the situation is even simpler when $q \in \{1,2\}$. Indeed, we clearly have $\mathbf{H}_0^{\{1\}}(x,y) = \frac{1}{1-x(1+y)}$ when $S = \{1\}$, and we obtain Equation (2) when applying Proposition 3.16 for $S = \{2\}$.

Example 3.18. Let $q \geq 3$. Consider *q-multiple hyperchord diagrams*, for which $S = q\mathbb{N}^*$. The generating function of connected crossing-free q-multiple hyperchord diagrams satisfies the functional equation

$$\left(\mathbf{CH}_0^{q\mathbb{N}^*}(x,y) - x\right) x^{q} - \left(\mathbf{CH}_0^{q\mathbb{N}^*}(x,y)^2 + \mathbf{CH}_0^{q\mathbb{N}^*}(x,y) - x\right)^{q-2} P_q\big(\mathbf{CH}_0^{q\mathbb{N}^*}(x,y)\,, x, y\big) = 0,$$

where $P_q(C,x,y)$ is the polynomial of degree 5 given by

$$P_q(C,x,y) = C^5 + (-x+2)\,C^4 + (xy-4x+1)\,C^3 + x(2x+y-3)\,C^2 + 2x^2(-y+2)\,C + x^3(-2+y).$$

The cases $q = 1$ and $q = 2$ are similar and left to the reader.

We now analyze the singular behavior of crossing-free hyperchord diagrams in $\mathcal{H}^S$. In this case, the argument is somehow indirect.

Proposition 3.19. *Let S be a non-empty subset of $\mathbb{N}^*$ different from the singleton $\{1\}$. The univariate generating function $\mathbf{H}_0^S(x,1)$ of crossing-free hyperchord diagrams of $\mathcal{H}^S$ has a unique smallest singularity ρ_S, and a square-root type singular expansion*

$$\mathbf{H}_0^S(x,1) \underset{x\sim\rho_S}{=} \alpha_S - \beta_S\sqrt{1 - \frac{x}{\rho_S}} + O\left(1 - \frac{x}{\rho_S}\right)$$

in a domain dented at $x = \rho_S$.

Proof. We start proving that $\mathbf{CH}_0^S(x,1)$ diverges at a finite value of $x = \varrho_S$. Consider an element $q \geq 2$ of S. Let $\varrho_{\{q\}}$ be the (unique) smallest singularity of $\mathbf{CH}_0^{\{q\}}(x,1)$. As discussed in Example 3.17, the generating function $\mathbf{CH}_0^{\{q\}}(x,1)$ has a square-root type singularity at $x = \varrho_{\{q\}}$, hence $\frac{d}{dx}\mathbf{CH}_0^{\{q\}}\big(\varrho_{\{q\}},1\big)$ diverges. Next, observe that

$$[x^n]\,\mathbf{CH}_0^S(x,1) \geq [x^n]\,\mathbf{CH}_0^{\{q\}}(x,1)$$

for all values of n, because all q-uniform hyperchord diagram are hyperchord diagrams of $\mathcal{H}^S$. Consequently,

$$[x^n]\,\frac{d}{dx}\mathbf{CH}_0^S(x,1) \geq [x^n]\,\frac{d}{dx}\mathbf{CH}_0^{\{q\}}(x,1)$$

for all values of n. Finally, as both functions are analytic at $x = 0$, there exists a real value $\varrho_S \leq \varrho_{\{q\}}$ such that $\frac{d}{dx}\mathbf{CH}_0^S(\varrho_S,1)$ diverges when x tends to ϱ_S as x grows from 0. In particular, ϱ_S

must be the dominant singularity of $\mathbf{CH}_0^S(x,1)$ (observe that we do not assure that the singularity is of square root-type, only that the derivative diverges at that point).

We continue analyzing the singular development of $\mathbf{H}_0^S(x,1)$ around its smallest singularity. Observe that Equation (10) can be written in the form $\chi(x\mathbf{H}_0^S(x,1)) = x$, where

$$\chi(u) = \frac{u}{1+\mathbf{CH}_0^S(u,1)}$$

is the functional inverse of $x\mathbf{H}_0^S(x,1)$. Let ρ_S be the dominant singularity of $x\mathbf{H}_0^S(x,1)$ (and hence the dominant singularity of $\mathbf{H}_0^S(x,1)$). Write $\tau_S = \rho_S\mathbf{H}_0^S(\rho_S,1)$. In particular, ρ_S satisfies that $\rho_S = \chi(\tau_S)$. Developing the relation $\chi'(\tau_S) = 0$, we obtain that

$$1+\mathbf{CH}_0^S(\tau_S,1) = \tau_S\frac{d}{dx}\mathbf{CH}_0^S(\tau_S,1)\,. \tag{11}$$

As $\mathbf{CH}_0^S(x,1)$ is analytic at $x=0$ and diverges at $x=\varrho_S$, Equation (11) has a solution $\tau_S < \varrho_S$. We have then that χ has a branch point at $u=\tau_S$, and by the Inverse Function Theorem $\mathbf{H}_0^S(x,1)$ ceases to be analytic at $x=\rho_S$. Finally, we conclude that $x\mathbf{H}_0(x,1)$ has a square root type singular behaviour around $x=\rho_S$. $\square$

In this particular setting, ρ_S is a computable constant that can be calculated (with a desired precision) in the following way. Observe that both $\mathbf{CH}_0^S(x,1)$ and $\frac{d}{dx}\mathbf{CH}_0^S(x,1)$ are analytic functions at $x=\tau_S$. Hence we can obtain approximations for τ_S by truncating conveniently the Taylor expansions of $\mathbf{CH}_0^S(x,1)$ and $\frac{d}{dx}\mathbf{CH}_0^S(x,1)$ in Equation (11). In fact, one needs to consider a lot of Taylor coefficients in order to obtain a good estimate of τ_S, because experimentally the position of the solution of Equation (11) is very close to the singularity of $\mathbf{CH}_0^S(x,1)$.

Once an approximation of τ_S is computed, we obtain a good approximation of both ρ_S and α_S using the relation $\rho_S = \chi(\tau_S)$ and $\tau_S = \rho_S\mathbf{H}_0(\rho_S,1) = \rho_S\alpha_S$. Finally, an approximation for β_S can be obtained using indeterminate coefficients on the relation $\chi(x\mathbf{H}_0^S(x,1)) = x$.

We have applied this method to approximate the constants τ_S, ρ_S, ${\rho_S}^{-1}$, α_S, and β_S for both q-uniform hyperchord diagrams (*i.e.* $S=\{q\}$, see Example 3.17) and q-multiple hyperchord diagrams (*i.e.* $S=q\mathbb{N}^*$, see Example 3.18), for $3\le q\le 7$. The results are shown in Table 2.

	S	τ_S	ρ_S	${\rho_S}^{-1}$	α_S	β_S
q-uniform	$\{3\}$	0.16648974	0.14078101	7.10323062	1.18261501	0.04374341
	$\{4\}$	0.29124158	0.22185941	4.50735894	1.31273036	0.08298341
	$\{5\}$	0.38048526	0.27126972	3.68636788	1.40260866	0.10797005
	$\{6\}$	0.44765569	0.30473450	3.28154504	1.46900231	0.12399216
	$\{7\}$	0.50026001	0.32902575	3.03927574	1.52042812	0.13445024
q-multiple	$3\mathbb{N}^*$	0.16334708	0.13864031	7.21290960	1.17820771	0.03365135
	$4\mathbb{N}^*$	0.28781764	0.22003286	4.54477579	1.30806666	0.05948498
	$5\mathbb{N}^*$	0.37742727	0.26987181	3.70546302	1.39854280	0.07361482
	$6\mathbb{N}^*$	0.44503426	0.30365836	3.29317462	1.46557553	0.08138694
	$7\mathbb{N}^*$	0.49802658	0.32817932	3.04711459	1.51754407	0.08564296

TABLE 2. Approximate values of the constants τ_S, ρ_S, ${\rho_S}^{-1}$, α_S, and β_S for the families of q-uniform and q-multiple hyperchord diagrams, for $3\le q\le 7$.

Observe that the growth constant for q-uniform hyperchord diagrams is just slightly smaller than the growth constants of the corresponding q-multiple hyperchord diagrams.

We have now obtained the complete asymptotic behavior of crossing-free hyperchord diagrams in $\mathcal{H}^S$ and we can therefore proceed to study hyperchord diagrams of $\mathcal{H}^S$ with precisely k crossings. Applying once more the same method as in Section 2.2, we obtain an expression of the generating function $\mathbf{H}_k^S(x,y)$ of hyperchord diagrams of $\mathcal{H}^S$ with k crossings in terms of the corresponding k-core hyperchord diagram polynomial

$$\mathbf{KH}_k^S(\mathbf{x},y) := \sum_{\substack{K\ k\text{-core}\\ \text{hyperchord}\\ \text{diagram of } \mathcal{H}^S}} \frac{\mathbf{x}^{\mathbf{n}(K)}\, y^{m(K)}}{n(K)},$$

and of the polynomials

$$\mathbf{H}_0^{S\,|\,n}(y) := [x^n]\,\mathbf{H}_0^S(x,y) \qquad \text{and} \qquad \mathbf{H}_0^{S\,|\,\le p}(x,y) := \sum_{n\le p} \mathbf{H}_0^{S\,|\,n}(y)\,x^n.$$

Proposition 3.20. *For any $k \ge 1$, the generating function $\mathbf{H}_k^S(x,y)$ of the hyperchord diagrams with k crossings and where the size of each hyperchord belongs to S is given by*

$$\mathbf{H}_k^S(x,y) = x\frac{d}{dx}\mathbf{KH}_k^S\left(x_{0,j} \leftarrow \frac{\mathbf{H}_0^{S\,|\,j}(y)}{x^j},\ x_{i,j} \leftarrow \frac{x^i}{(i-1)!}\frac{d^{i-1}}{dx^{i-1}}\frac{\mathbf{H}_0^S(x,y)-\mathbf{H}_0^{S\,|\,\le i+j}(x,y)}{x^{i+j+1}},\ y\right).$$

In particular, $\mathbf{H}_k^S(x,y)$ is a rational function of the generating function $\mathbf{H}_0^S(x,y)$ and of the variables x and y.

Using this composition scheme and the singular behavior of $\mathbf{H}_0^S(x,1)$ described in Proposition 3.19, computations similar to that of the proof of Proposition 3.9 lead to the following asymptotic result.

Proposition 3.21. *Let $k \ge 1$ and let S be a non-empty subset of $\mathbb{N}^*$ different from the singleton $\{1\}$. Let ρ_S be the smallest singularity and α_S, β_S be the coefficients of the asymptotic expansion of the generating function $\mathbf{H}_0^S(x,1)$ around ρ_S, as defined in Proposition 3.19. Let $\Psi(k,S)$ denote the maximum value of the potential function*

$$\psi(K) := \sum_{\substack{i\ge1\\ j\ge0}} (2i-3)\,n_{i,j}(K)$$

over all k-core hyperchord diagrams of $\mathcal{H}^S$. There is a constant Λ_S such that the number of hyperchord diagrams with k crossings, n vertices, and where the size of each block belongs to S is

$$[x^n]\,\mathbf{H}_k^S(x,1) \underset{\substack{n\to\infty\\ \gcd(S)|n}}{=} \Lambda_S\, n^{\frac{\Psi(k,S)}{2}}\, {\rho_S}^{-n}\,(1+o(1)),$$

for n multiple of $\gcd(S)$, while $[x^n]\,\mathbf{H}_k^S(x,1) = 0$ if n is not a multiple of $\gcd(S)$. More precisely, the constant Λ_S can be expressed as

$$\Lambda_S := \frac{\gcd(S)\,\rho_S\,\Psi(k,S)}{\Gamma\big(\frac{\Phi(k,S)}{2}+1\big)} \sum_K \frac{1}{n(K)} \prod_{j\ge0}\left(\frac{\mathbf{D}_0^{S\,|\,j}(1)}{{\rho_S}^j}\right)^{n_{0,j}(K)} \alpha_S^{n_{1,j}(K)} \prod_{\substack{i\ge1\\ j\ge0}}\left(\frac{\beta_S\,(2i-5)!!}{2^{i-1}\,{\rho_S}^{i+j}\,(i-1)!}\right)^{n_{i,j}(K)},$$

where we sum over the k-core hyperchord diagrams K of $\mathcal{H}^S$ which maximize the potential function $\psi(K)$.

Remark 3.22. As for the partitions $\mathcal{P}^S$, we observe that the exponent $\frac{\Psi(k,S)}{2}$ in the polynomial growth of $[x^n]\,\mathbf{H}_k^S(x,1)$ is not a constant of the class, but really depends on the value of both S and k. The reader is invited to work out examples of q-uniform and q-multiple hyperchord diagrams to get convinced that this exponent can have an unexpected behavior.

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